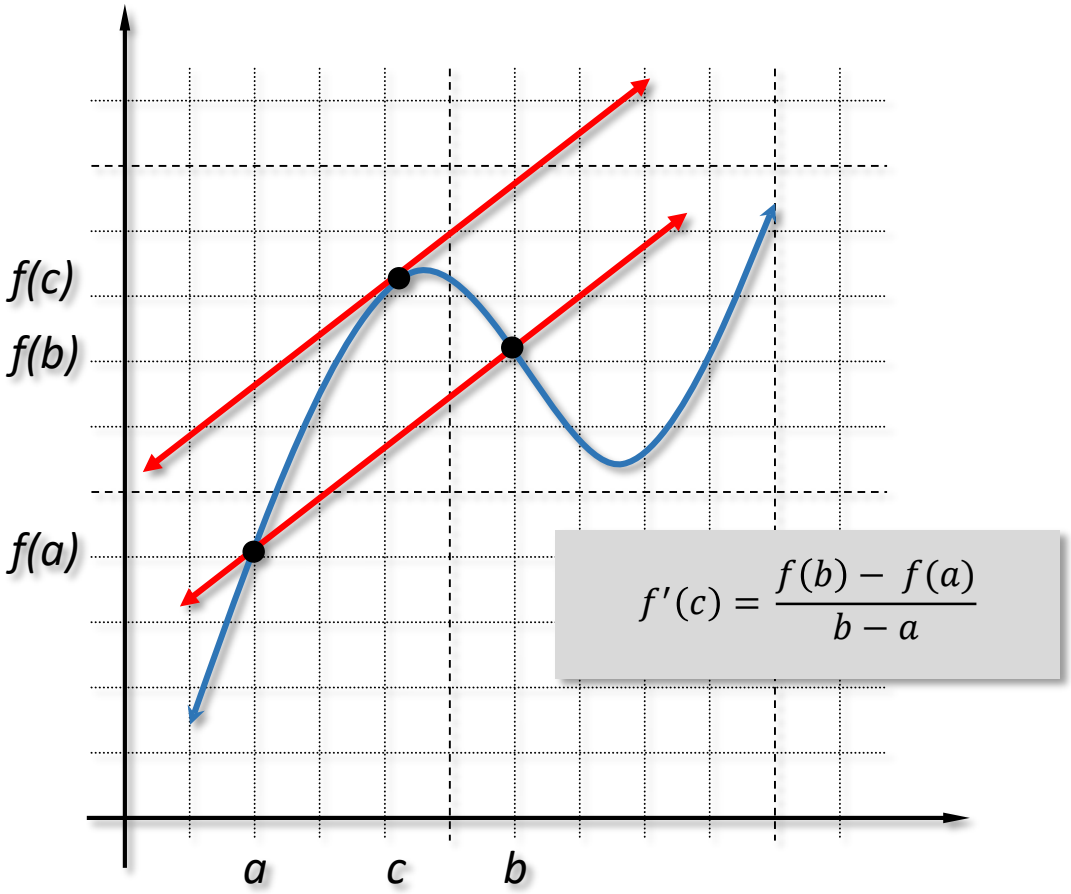

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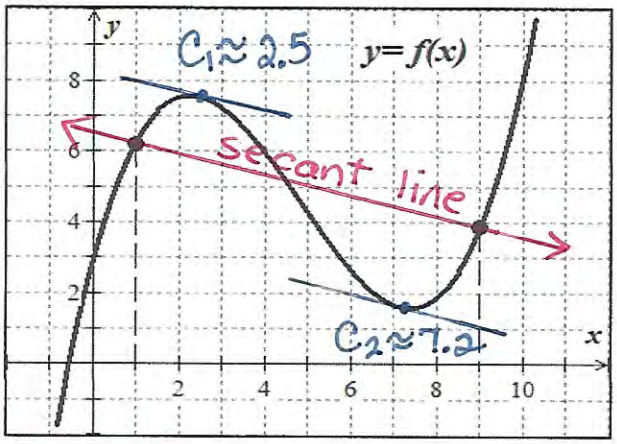
UNIT 5:

ANALYTICAL APPLICATIONS OF DIFFERENTIATION NOTES

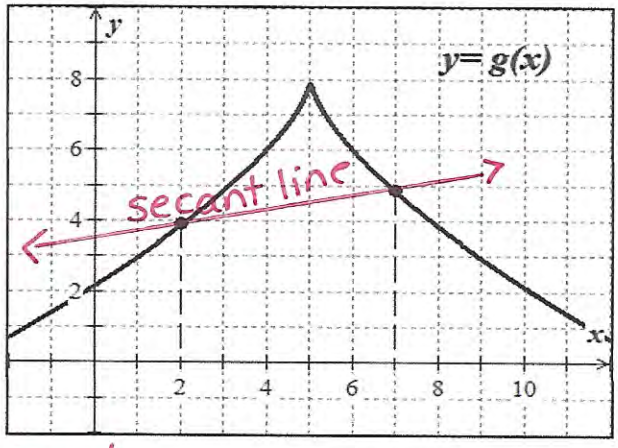
Lesson 1: Mean Value Theorem

Topic 5.1: Using the Mean Value Theorem

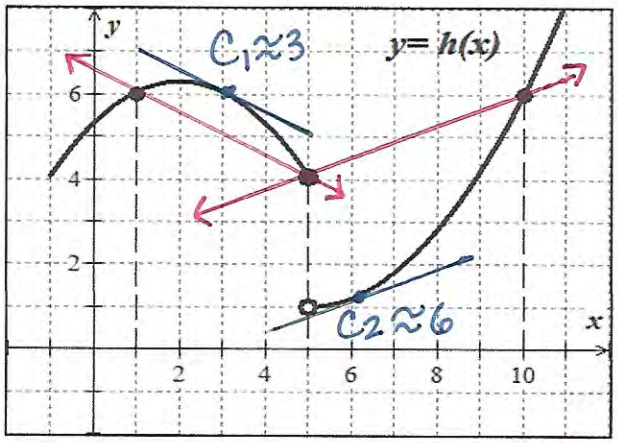
DISCOVERY: Use the graphs and questions below to explore conclusions to the Mean Value Theorem.



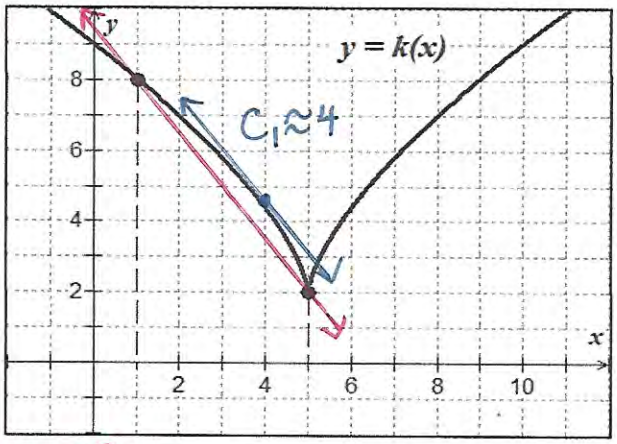
$m = -\frac{1}{4}$ *Figure 1*



$m = \frac{1}{5}$ *Figure 2*



$m_1 = -\frac{1}{2}$ *Figure 3* $m_2 = \frac{2}{5}$



$m = -\frac{3}{2}$ *Figure 4*

1. Draw the secant lines for each figure on the given intervals.

2. Are there any points on the graph of each function where the tangent line to the graph appears to be parallel to the secant line you've drawn? Draw a tangent line(s) between the endpoints of the given interval, if possible. Let this point of tangency be called $x = c$, for each figure.

3. For each figure, estimate the value of $x = c$ where the tangent line is parallel to the secant line; or, explain why it is not possible.

1) $C_1 \approx 2.5$, $C_2 \approx 7.2$ 2) not possible cusp 3) $C_1 \approx 3$, $C_2 \approx 6$ 4) $C_1 \approx 4$

4. Which functions are continuous on the given intervals?

f(x), g(x), k(x)

5. Which functions are differentiable on the given intervals?

f(x) k(x)

6. Which are not differentiable? Explain why.

g(x) has a cusp ; h(x) is not continuous at x = 5

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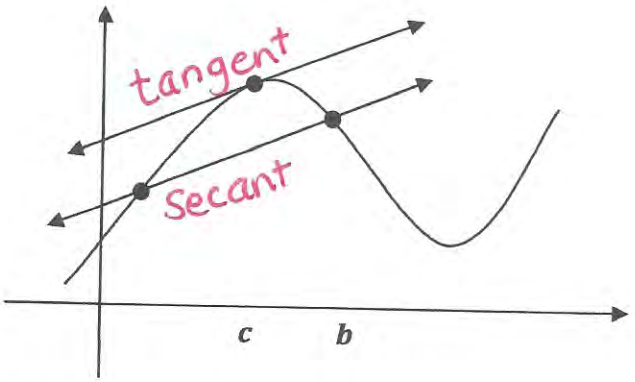
Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

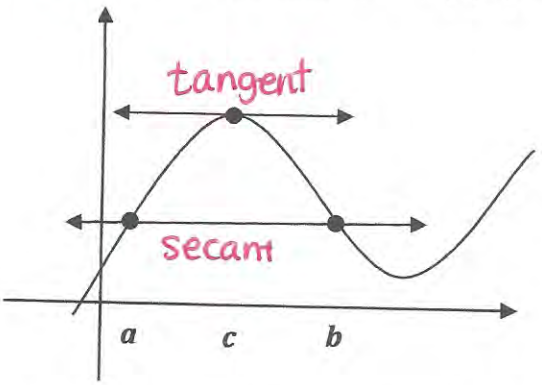


***VERBALLY SAYS*:** Instantaneous rate of change equals average rate of change. *

***GRAPHICALLY SAYS*:** Tangent line is parallel to the secant line.

Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there exists at least one number c in (a, b) where $f'(c) = 0$.



If these conditions hold true, then there is **at least** one number between a and b so that the **tangent line is horizontal**.

Important Note: Both Rolle's Theorem and the MVT guarantee at least one value strictly on the OPEN interval.

What is the difference in conditions between the MVT and Rolle's Theorem?

Rolle's has a secant line whose slope is zero

What does Rolle's Theorem tell you must exist on a given interval?

So, an extrema (max or min) must exist.

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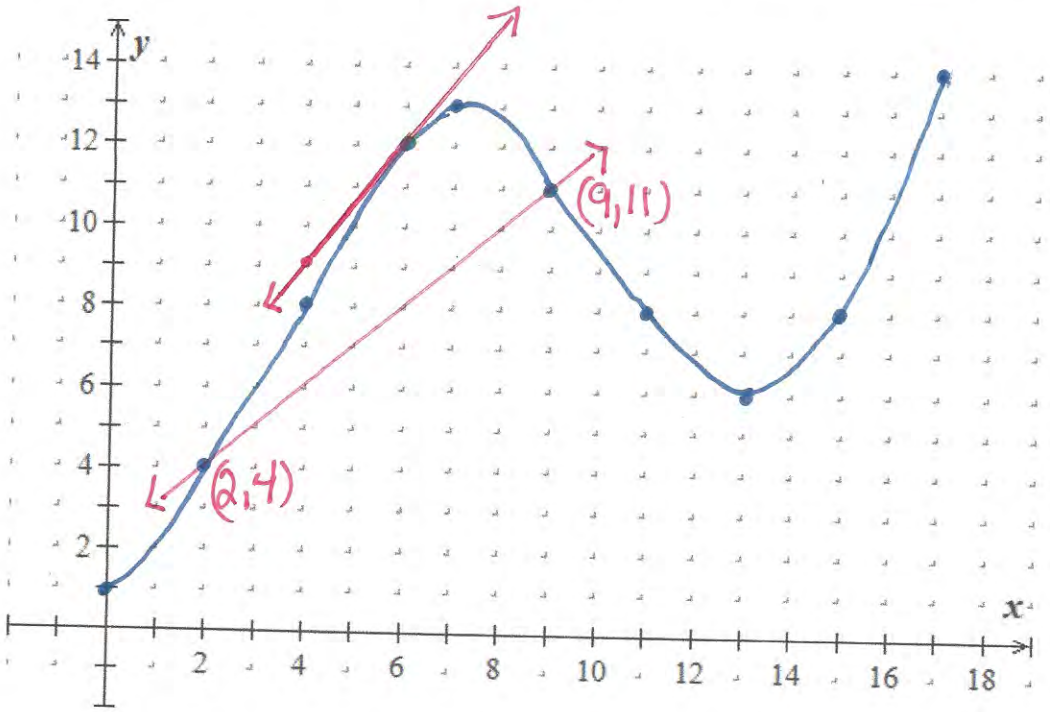
EX. #1: Use the table of values below to sketch the differentiable continuous function on the grid.

x	0	2	4	6	7	9	11	13	15	17
$g(x)$	1	4	8	12	13	11	8	6	8	14



A. Find the average rate of change of $g(x)$ on the interval $[2, 9]$.

$$AROC = \frac{f(9)-f(2)}{9-2}$$

$$= \frac{11-4}{7}$$

$$AROC = 1$$

B. Is $g'(6)$ equal to the slope of the secant line from part A? Justify.

Using $(6,12)$ and $(4,9)$

$$g'(6) \approx \frac{12-9}{6-4} = \frac{3}{2}$$

$\frac{3}{2} \neq 1$, no!

C. Does Rolle's Theorem apply on the interval $[4,9]$? Explain why or why not.

No, because $f(4) \neq f(9)$

So $\frac{f(9)-f(4)}{9-4} \neq 0$

D. Is there another interval where Rolle's Theorem applies? Justify.

Yes, Rolle's Theorem can be applied on the following intervals $4 \leq x \leq 11$ and $11 \leq x \leq 15$, in both intervals m_{sec} is zero, g is continuous + differentiable

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EX#2: Let f be the function given by $f(x) = x^2 + 2x - 3$. Find the number c that satisfies the conclusion of Rolle's Theorem for f on $[-3, 1]$.

1) $f'(x) = 2x + 2$

$2(x+1) = 0$

$\underline{\underline{x = -1}}$

2) f is continuous + differentiable (polynomial)

3) $f(-3) = f(1) = 0$

4) $\dfrac{f(1) - f(-3)}{1-(-3)} = 2x + 2$

when $x = -1$

5) $\therefore$ $\underline{\underline{c = -1}}$

EX #3: A sports car speedometer registers 50 miles per hour as it passes a patrol car stationed at a mileage marker. Four minutes later, the car passes another patrolman at a marker that is five miles from the first. The driver notes that the speedometer registers 55 miles per hour. Can the highway patrol use the Mean Value Theorem to prove that the driver was exceeding the speed limit of 70 miles per hour at some time between the two mileage markers?

A $\quad s(t) \quad$ B

$t=0 \qquad t = 1/15$

Let t = elapsed time, in hours after car passes the first patrol car.

$s(t)$ = distance, miles from first marker at time, t.

$t = \frac{4}{60} = \frac{1}{15}$ hour

$V_{avg} = \dfrac{s(\frac{1}{15}) - s(0)}{\frac{1}{15} - 0} = \dfrac{5}{\frac{1}{15}}$

$\underline{\underline{V_{avg} = 75 \text{ mi/hrs}}}$

Yes, by MVT car was traveling at 75mph at least once on 0 to 4 minutes

EX #4: Let $h(x) = 5 - \frac{4}{x}$, find all values of c in the open interval $(1, 4)$ guaranteed by the Mean Value Theorem.

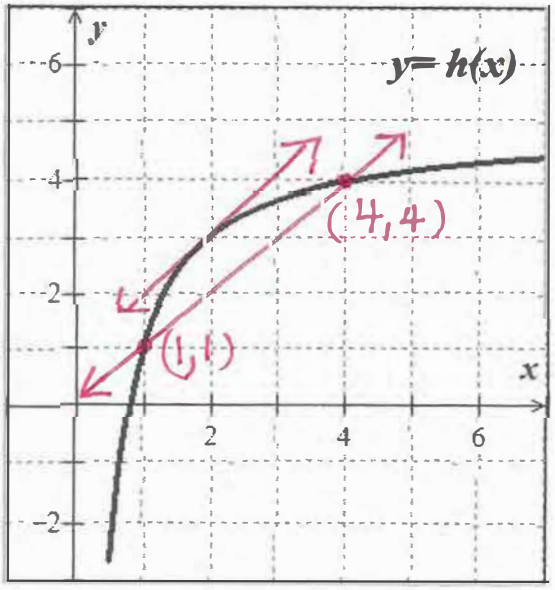
$\dfrac{h(4) - h(1)}{4-1} = \dfrac{4}{x^2}$

$1 = \dfrac{4}{x^2}$

$x^2 = 4$

$x = \pm 2$

$\therefore$ $c = 2$ satisfies MVT on $1 < x < 4$.



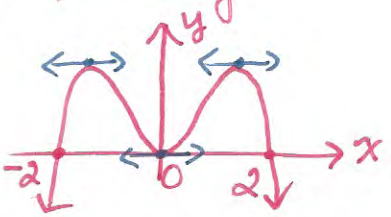
M.V.T. allows us to identify exactly where graphs rise or fall:

Functions with positive / negative derivatives are increasing / decreasing functions.

EX#5: Find all values between the x-intercepts for which Rolle's Theorem holds for the roots of $f(x) = 4x^2 - x^4$. Verify with your calculator.

1) $f(x)$ is continuous & differentiable everywhere

Graphically:



2) $f(x) = 0$

$-x^2(x^2-4)=0$

$x = 0, 2, -2$

3) $f'(x) = 8x - 4x^3$

$-4x(x^2-2)=0$ ← critical numbers

$x = 0, \sqrt{2}, -\sqrt{2}$

Conclusion:

For the interval $-2<x<2$, Rolle's Theorm applies for $x=-\sqrt{2}$ $x=0$ and $x=\sqrt{2}$.

EX #6: Determine whether $g(x) = \sin 2x + 2x$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, \pi]$. If so, find all numbers c in (a, b) such that $g(b) - g(a) = g'(c)(b-a)$

1) Sine and polynomials are continuous and differentiable.

2) $g(0)=0$

$g(\pi)=2\pi$

3) $g'(x) = 2\cos 2x + 2$

$\frac{g(\pi)-g(0)}{\pi - 0} = g'(x)$

$\frac{2\pi - 0}{\pi - 0} = 2(\cos 2x + 1)$

$2 = 2(\cos 2x + 1)$

$\cos 2x + 1 = 1$

$\cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$ on $[0,\pi]$

4) $\therefore c = \frac{\pi}{4}, \frac{3\pi}{4}$ on $0 \le x \le \pi$ to satisfy M.V.T.

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Lesson 2: Extrema on an Interval

Topic 5.2: Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

The maximum and minimum values (max and min) of a function are called the **extreme values** or **extrema** (singular: extremum) of a function. The process used to finding them is referred to as **optimization**. There will be times when we want to find the max and min for x on the entire domain, and at other times on a particular interval. In this lesson, we will learn how to use calculus to find extrema, rather than a calculator.

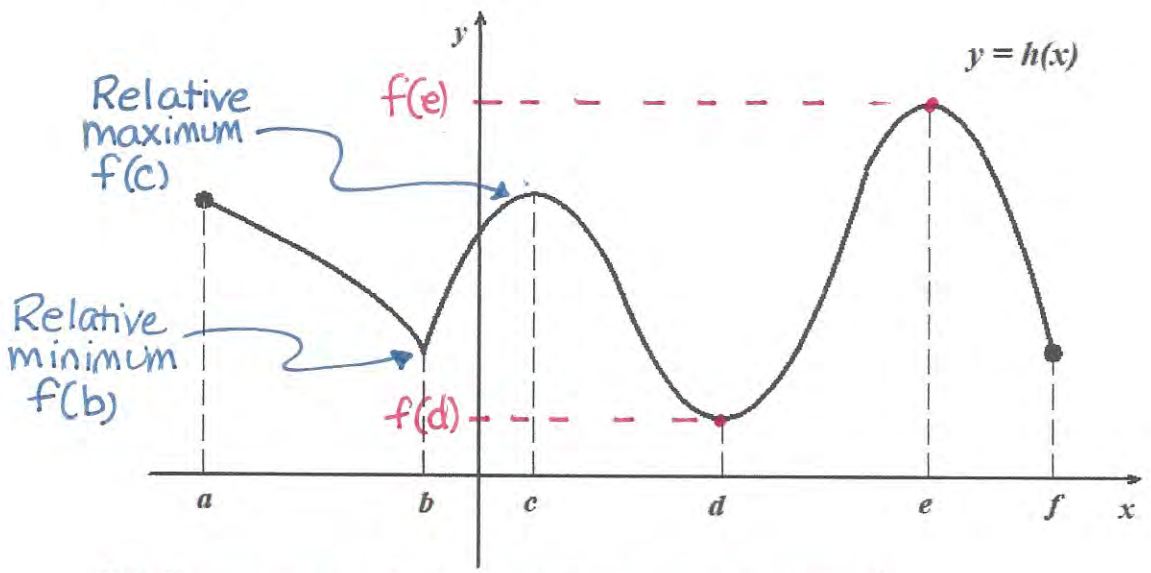
Definition: (Absolute/Global) Extrema

If f is a function on an interval I, then $y = f(c)$ is the

I. **(Absolute/Global) Maximum** on I, if and only if $f(c) \geq f(x)$ for all x in I.

II. **(Absolute/Global)Minimum** on I, if and only if $f(c) \leq f(x)$ for all x in I.

EX #1: The graph of $y = h(x)$ is shown below. Determine the extrema of $h(x)$ on the interval $a \leq x \leq f$.



f(d) is absolute minimum at x = d

f(e) is absolute maximum at x = e

Defining Relative and Absolute Extrema of a Function

Relative or Local Extrema	y-values on the graph of a function where the function changes from increasing to decreasing and vice versa.
Absolute or Global Extrema	the highest or lowest y-values on the graph of a function or on a specified domain of a function.

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Definition of Relative/Local Extrema

1. If there is an open interval containing $\boldsymbol{c}$ on which $\boldsymbol{f(c)}$ is a maximum, then $\boldsymbol{f(c)}$ is called a **relative maximum of** $\boldsymbol{f}$.

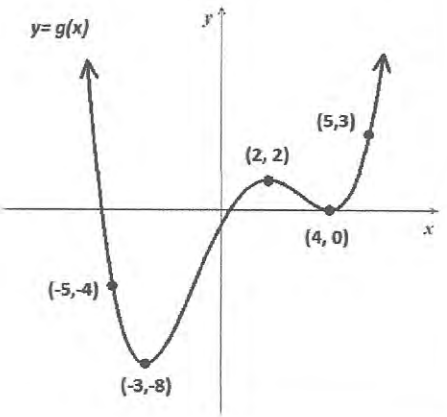
2. If there is an open interval containing $\boldsymbol{c}$ on which $\boldsymbol{f(c)}$ is a minimum, then $\boldsymbol{f(c)}$ is called a **relative minimum of** $\boldsymbol{f}$.

NOTEWORTHY THOUGHTS:

- ❑ *When we say an "open interval containing c" we mean that we can find some interval (a, b), not including the endpoints, such that $a < c < b$.*
- ❑ *That is, c will be contained somewhere inside the interval and will not be either of the endpoints.*
- ❑ *A relative extrema is slightly different than an absolute extrema.*
- ❑ *All that's necessary for a point to be a relative max or min is for that point to be a maximum or minimum in some interval of x's around $x = c$.*
- ❑ *There can be smaller or larger values of the function at some other location, but relative to $x = c$, $f(c)$ is smaller or larger than all the other function values in the "neighborhood of c."*

In many cases, we will be asked to analyze both open interval and closed interval functions.

EX #2: Consider the graph of $y = g(x)$, shown at right.



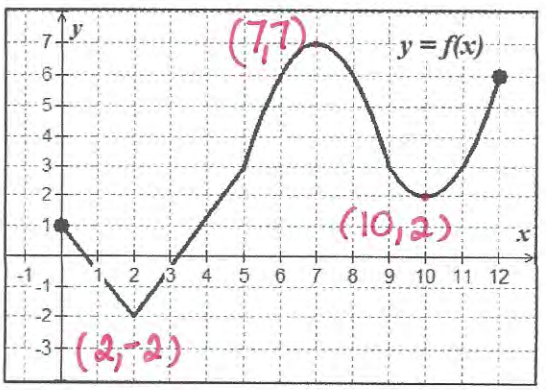
A. State the relative extrema of $g(x)$

rel. max is 2 at $x=2$
rel min is 0 at $x=4$

B. Considering the domain for g, what are the absolute extrema of the function?

abs. max does not exist b/c $g(x)$ is unbounded above
abs. min is -8 at $x=-3$

EX #3: Consider the graph of $y = f(x)$ on $0 \le x \le 12$, shown below.



A. Find the relative extrema of $f(x)$.

rel. max is 7 at $x=7$
rel min is 2 at $x=10$
rel. min is -2 at $x=2$

B. Considering the domain for f, what are the absolute extrema of the function?

abs. max is 7 at $x=7$
abs min is -2 at $x=2$

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What conditions would be necessary in order to have BOTH an ABSOLUTE MAXIMUM and an ABSOLUTE MINIMUM?

The Extreme Value Theorem (EVT)

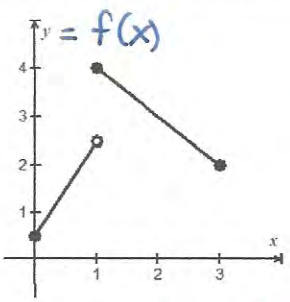
If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval $[a, b]$.

Extreme values can occur at interior points or endpoints of an interval. A function may have both maximum and minimum values over an interval, either a maximum or a minimum, or no extrema on an interval. Let's explore some examples below.

When the hypothesis (or "IF" statement) is not met, either continuity or the closed interval, there is no guarantee of the conclusion ("THEN" statement). However, a maximum, minimum, or both may still exist, there is just no guarantee. Let's explore a few examples where this happens.

EX #4: For each graph below, explicitly state where the hypothesis of the EVT fails on the interval $[0, 3]$. Then, determine if the function has an extrema on the interval.

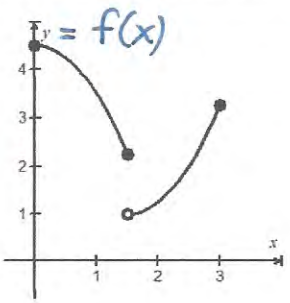
A.



EVT does not apply b/c f is not continuous on [0,3]

abs. min at x=0
abs. max at x=1

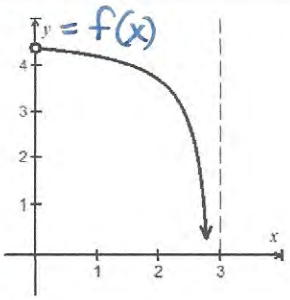
B.



EVT fails f is not continuous on [0, 3] at x = 1.5

absolute max at x = 0
no abs. min

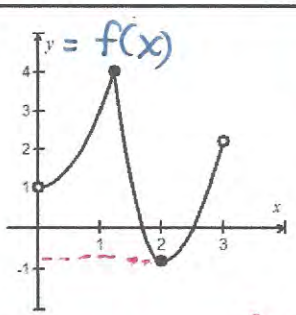
C.



EVT does not apply since f is not continuous on closed interval [0,3]

no extrema

D.



EVT fails since f is not continuous on [0,3]

abs. max of 4 at x = 1
abs min at x = 2

The Extreme Value Theorem gives us the knowledge of ***WHEN* extrema** occur. But how will we find them algebraically when given an equation only, without the benefit of a graph? So our next question should be:

How can we identify all the values of x where extreme values occur?

EX #5: For each function shown below, justify (explain) why the Extreme Value Theorem (EVT) can be applied, or why it does not apply on the given interval.

A. $f(x) = \dfrac{2x-3}{x-2}$ **Interval:** $-4 \le x \le 4$	EVT does not apply for f(x) on $-4 \le x \le 4$ because f is not continuous at $x=2$, which is contained on the interval.
B. $g(x) = e^{x-2} + 1$ **Interval:** $-5 \le x \le 5$	g(x) is continuous for all reals so EVT applies for g(x) on $-5 \le x \le 5$
C. $h(x) = x\sqrt{x+2}$ **Interval:** $-6 \le x \le 6$	Since h(x) is continuous only for values $x \ge -2$, EVT does not apply for h(x) on $-6 \le x \le 6$.

Definition: Critical Number (Value)

A **critical number (value)** of a function f is an x-value, $x = c$, **in the domain** of f such that either

$f'(c) = 0$ or $f'(c)$ DOES NOT EXIST

If $x = c$ is a critical value, then $(c, f(c))$ is called a **critical point**.

Theorems:

Relative/Local Extrema can only occur at a critical value on an OPEN interval.

(Absolute/Global) Extrema must occur at a critical number OR at an ENDPOINT of an interval.

Topic 5.5: Using the Candidates Test to Determine Absolute (Global) Extrema

Now we need to put these ideas together with a little direct practice. Let's develop an algebraic approach to what will be called ***THE CANDIDATES TEST*** for finding extrema.

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EX #6: Find the absolute extrema of the function on the given interval, provided the EVT is applicable. If it is not, justify why.

A. $f(x) = 8x^3 - 3x^2 - 9x + 2$ on $[-1, 2]$

$f'(x) = 24x^2 - 6x - 9$

$8x^2 - 2x - 3 = 0$

$(4x-3)(2x+1) = 0$

$x = 3/4 \quad x = -1/2$

Candidates Test:

x	-1	-0.5	0.75	2
f(x)	0	4.75	-3.063	36

min: -3.063 max: 36

B. $g(x) = x - 2\sin x$ on $[0, 2\pi]$

$g'(x) = 1 - 2\cos x$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

Candidates Test:

x	0	$\frac{\pi}{3}$	$\frac{5\pi}{3}$	2π
g(x)	0	-0.685	6.968	6.283

min: -0.685 max: 6.968

C. $h(x) = \sin^2 x - \cos x$ $\quad 0 \le x \le \frac{3\pi}{2}$

$h'(x) = 2\sin x \cos x + \sin x$

$\sin x(2\cos x + 1) = 0$

$x = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

Candidates Test:

x	0	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
h(x)	-1	1.25	1	1.25	1

min: -1 max: 1.25

D. $f(x) = ln\,(x^2 - 9)$ on $[-2, 5]$

f(x) is not continuous on [-2,5] since domain of f(x) is undefined from $-3 \le x \le 3$. So EVT does not apply.

E. $h(x) = \sqrt[3]{(x-2)^2}$ on $[1, 10]$

$h'(x) = \frac{2}{3\sqrt[3]{x-2}} = 0$

$h'(x) \neq 0$ and $h'(x)$ dne when $x = 2$

Candidates Test.

x	1	2	10
h(x)	1	0	4

min: 0 max: 4

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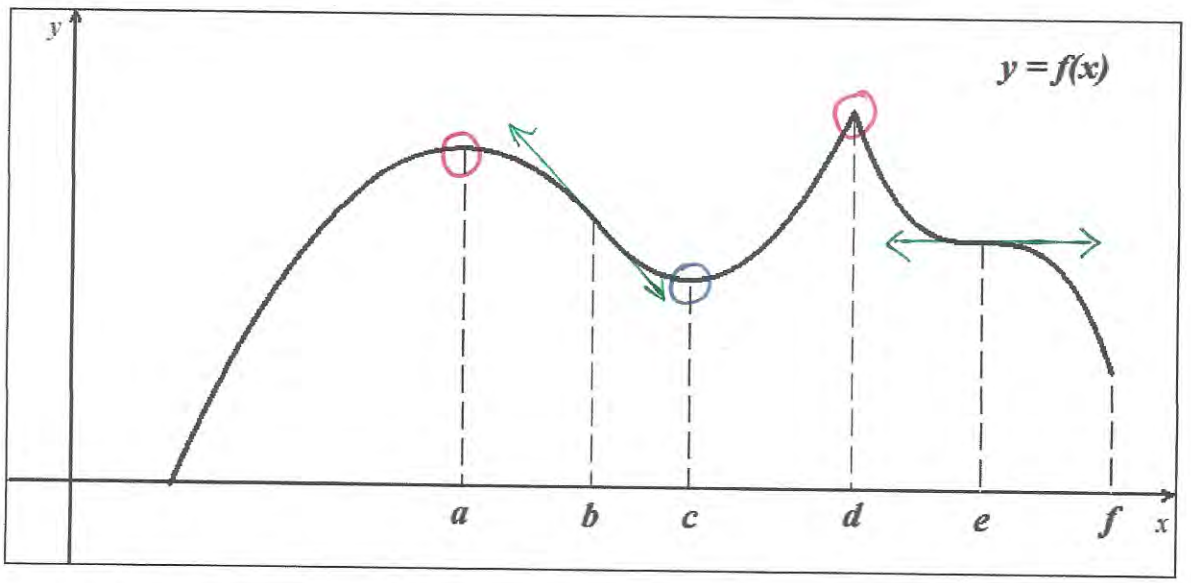
Lesson 3: The First Derivative Test
Increasing and Decreasing Functions

Topic 5.3: Determining Intervals on Which a Function Is Increasing or Decreasing

Derivatives can be used to classify relative extrema as either relative minima or relative maxima. As we begin to focus on analyzing functions, we can determine some predictable and specific behaviors for functions in our calculus journey through "the mathematics of change."

EX #1: ***This exploration leads to the conditions and outcomes of the first derivative test. Use the graph of the function f*** **to answer the questions below.**



1. Using a ***RED*** colored pencil, circle all the points corresponding to the local maxima of f. (Do not include endpoints.)

A. State the x-values where the local maxima of f occur. $x=a$; $x=d$

B. At each maximum value, f changes from increasing to decreasing

C. At each maximum value, f' changes from positive to negative.

2. Using a ***BLUE*** colored pencil, circle all the points corresponding to the local minima of f. (Do not include endpoints.)

A. State the x-values where the local minima of f occur. $x=c$

B. At each minimum value, f changes from decreasing to increasing

C. At each minimum value, f' changes from negative to positive

3. At each relative (local) extreme value, $f'(x)$, the derivative of f, is either zero or undefined (DNE)

4. Using a ***GREEN*** colored pencil, sketch a tangent line, if possible, at the following points and determine their sign.

A. At point b, $f'(b)$ is negative. Describe the change in the behavior of the graph of f at this value. concave down to concave up

B. At point d, $f'(d)$ is undefined. Describe the change in the behavior of the graph of f at this value. f goes from increasing to decreasing

C. At point e, $f'(e)$ is zero. Describe the change in the behavior of the graph of f at this value. concave up to concave down

D. Of these three points which two are points of inflection? $x=b$ and $x=e$

5. If a function f has a relative extremum at a point, then the derivative at the point is either zero or undefined. Is the converse of the statement also true? no

What point(s) justifies your answer? at $x=e$ $f'(x)=0$, but point "e" is not an extrema of f

Definitions: Increasing and Decreasing Functions

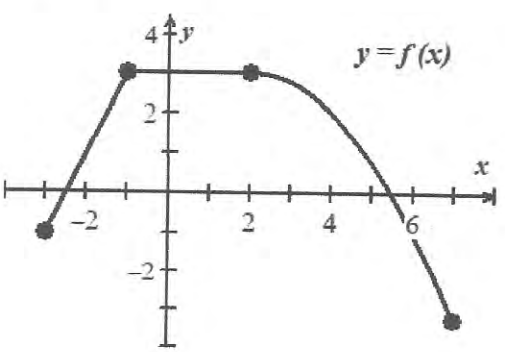
- ❑ A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- ❑ A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 > x_2$ implies $f(x_1) > f(x_2)$.

EX #2: The graph of $y = f(x)$, shown at right, is defined on $[-3, 7]$. List the *open intervals* over which the function is increasing, decreasing, or constant.

A. $f(x)$ is increasing on $-3<x<-1$

B. $f(x)$ is decreasing on $2<x<7$

C. $f(x)$ is constant on $-1<x<2$



NOTE: A function is said to be MONOTONIC on an interval if its first derivative (which need not be continuous) does not change sign.

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How can we find the values of x on the graph of a function where it is increasing, decreasing, or constant? Some important ideas come to mind here. If $f(x)$ is a **continuous function**, then the derivative can only change signs at a **critical value.** And, if $f(x)$ is a **discontinuous function**, then the derivative could change signs at a **critical value** or at a **discontinuity**.

How to Find Intervals of Increase and Decrease

Let f be a continuous function on the closed interval $[a, b]$. To find the intervals on which f is increasing or decreasing:

1. **Find the critical values** of f, including discontinuities, in the interval (a, b).
2. Create a **number line chart** using these critical numbers.
3. Choose ***x*-values in between the critical numbers** to test the sign of f' on your sign chart.

NOTE: Your sign chart is NOT A VALID JUSTIFICATION. You MUST write a clear and concise statement communicating your mathematical language related to sign changes of the derivative. ***DO NOT USE PRONOUNS!***

For example,
"The function g is increasing on the interval $(-2, 5)$ ***since*** $g'(x) > 0$***."***

Topic 5.3: Determining Intervals on Which a Function Is Increasing or Decreasing

EX #3: Find the open intervals on which $f(x) = \frac{2}{3}x^3 - x^2 - 4x + 2$ is increasing and or decreasing. Justify.

① $f'(x) = 2x^2 - 2x - 4$
$2(x^2 - x - 2) = 0$
$(x-2)(x+1) = 0$
$x = 2,\ x = -1$

② Sign chart

f'	$+++$	0	$---$	0	$+++$
f	$\nearrow$	-1	$\searrow$	2	$\nearrow$

③ $f(x)$ increases on $(-\infty, -1) \cup (2, \infty)$ because $f'(x) > 0$.
$f(x)$ decreases on $(-1, 2)$ because $f'(x) < 0$.

When presented with a continuous function, knowing ***that*** the sign of the derivative changes at a point $(c, f(c))$, as well as, knowing ***how*** the sign changes will give us insight into the existence of extrema.

Topic 5.4: Using the First Derivative Test to Determine Relative (Local) Extrema

FIRST DERIVATIVE TEST:

Let c be a critical number of a function f that is continuous on an open interval. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a ***relative minimum*** of f.
2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a ***relative maximum*** of f.
3. If $f'(x)$ does not change signs at c, then $f(c)$ is neither a ***relative minimum*** nor a ***relative maximum*** of f.

We will explore how to find points of inflection and intervals where a function is concave up or concave down in the next lesson. The following chart will be helpful in summarizing and ***using correct vocabulary*** when justifying conditions:

Function Behavior				
$f(x)$	$\cup$	$\cap$	$\nearrow$	$\searrow$
$f'(x)$	$\nearrow$	$\searrow$	$+$	$-$
$f''(x)$	$+$	$-$		

Symbol	Vocabulary
$+$	Positive
$-$	Negative
$\nearrow$	Increasing
$\searrow$	Decreasing
$\cup$	Concave Up
$\cap$	Concave Down

EX #4: Given $f(x) = x^4 - 8x^2 + 1$

A. Find the open intervals on which $f(x)$ is increasing and/or decreasing. Justify.

$f'(x) = 4x^3 - 16x$

$4x(x^2 - 4) = 0$

$x = 0, -2, 2$

f': $- - -$ 0 $+ + +$ 0 $- - -$ 0 $+ + +$

f: $\searrow$ -2 (min) $\nearrow$ 0 (max) $\searrow$ 2 (min) $\nearrow$

B. Determine the x-values of any local extrema of $f(x)$. Justify.

$f(x)$ increases where $f'(x) > 0$, on $(-2, 0) \cup (2, \infty)$

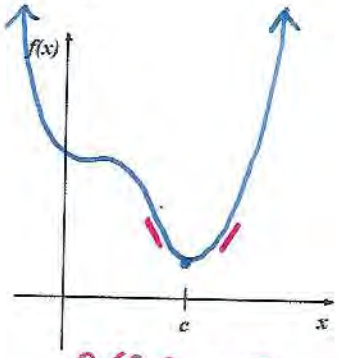
$f(x)$ decreases where $f'(x) < 0$, on $(-\infty, -2) \cup (0, 2)$

max occurs at $x = 0$ since $f'(x)$ changes from positive to negative here.

min occurs at $x = -2$ and $x = 2$, $f'(x)$ changes from negative to positive here.

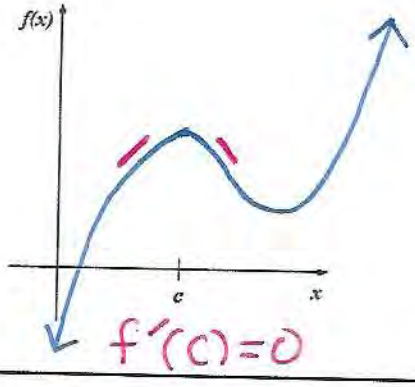
EX #5: For each statement below, sketch a function that represents the results of the first derivative test for a continuous function $y = f(x)$ with a critical number at $x = c$.

A. $f(x)$ has an absolute minimum at $x = c$, because $f'(x)$ changes sign from negative to positive at $x = c$.



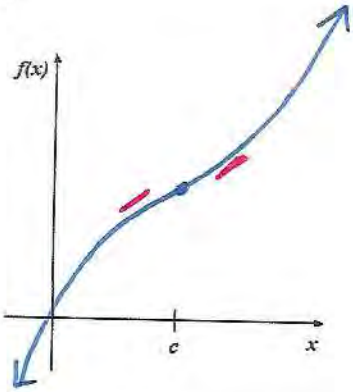
$f'(c) = 0$

B. $f(x)$ has a local maximum $x = c$, because $f'(x)$ changes sign from positive to negative at $x = c$.

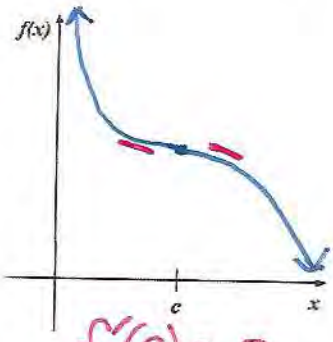


$f'(c) = 0$

C. $f(x)$ has no local extrema at $x = c$ because $f'(x) > 0$ for all $x > 0$ and is strictly monotonic.



D. Since $f'(x)$ is negative immediately on either side of $x = c$, the function $f(x)$ has neither a local maximum nor local minimum at $x = c$.



$f'(c) = 0$

EX #6: Find the exact values of any relative extrema of the function $g(x) = \frac{1}{2}x - \sin x$ on the interval $[0, 2\pi]$. Justify.

$g'(x) = \frac{1}{2} - \cos x$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

g'		$- -$	0	$+ +$	0	$- -$	
	0	$\searrow$	$\pi/3$	$\nearrow$	$\frac{5\pi}{3}$	$\searrow$	2π
g			min.		max.		

① g has a rel. min. value of $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$ at $x = \frac{\pi}{3}$ since $g'(x)$ changes from positive to negative

② g has a rel max. value of $\frac{5\pi}{6} + \frac{\sqrt{3}}{2}$ at $x = \frac{5\pi}{3}$ since $g'(x)$ changes from negative to positive

ANSWER THE QUESTION BEING ASKED . . . Are you looking for extrema (Y-VALUES) or the location where the extrema occurs (X-VALUES)?

EX #7: Find the relative extrema for $f(x) = (x^2 - 4)^{\frac{2}{3}}$. Justify.

$f'(x) = \frac{2}{3}(x^2-4)^{-1/3}(2x)$

$f'(x) = \frac{4x}{3\sqrt[3]{x^2-4}}$

$4x = 0$ $\quad$ $x^2 - 4 \neq 0$

$x = 0$ $\quad$ $x = -2, 2$

critical #

$x = 0$ $\quad f' = 0$

$x = \pm 2$ $\quad f'$ dne

f'	$- -$	dne	$+ +$	0	$- -$	dne	$+ +$	
f	↘	-2 min	↗	0 max	↘	2 min	↗	

① f has rel. max of $\sqrt[3]{16}$ or $2\sqrt[3]{2}$ at $x = 0$ b/c f' changes sign from positive to negative.

② f has rel. min of 0 at $x = -2$ and $x = 2$ b/c f' changes sign from negative to positive.

EX #8: Find the x-coordinates of the relative extrema for $h(x) = \sqrt[3]{x^2}\,(x-2)$

$h'(x) = \frac{2}{3}x^{-1/3}(x-2) + x^{2/3}$

$= \frac{2(x-2)+3x}{3x^{1/3}}$

$= \frac{5x-4}{3x^{1/3}}$

$x = \frac{4}{5}$ $\quad$ $x = 0$

$h' = 0$ $\quad$ dne

h'	$+ +$	dne	$- -$	0	$+ +$	
h	↗	0	↘	$4/5$	↗	

rel. max occurs at $x = 0$

rel. min occurs at $x = 4/5$

EX #9: Given $g(x) = \frac{x^2-3}{x-2}$, **(i)** Find the intervals of increasing/decreasing and **(ii)** Fnd the x-coordinates of the relative extrema.

$g'(x) = \frac{(x-2)(2x) - (x^2-3)}{x-2}$

$g'(x) = \frac{x^2 - 4x + 3}{(x-2)^2}$

$g'(x) = \frac{(x-3)(x-1)}{(x-2)^2}$

g'	$+ +$	0	$- -$	dne	$- -$	0	$+ + +$		
g	↗	1 max	↘	2	↘	3 min	↗		

(i) g increases $(-\infty, 1) \cup (3, \infty)$ where $g'(x) > 0$

g decreases $(1, 2) \cup (2, 3)$ where $g'(x) < 0$

(ii) rel. max occurs at $x = 1$

g' changes from (+) to (−)

rel. min occurs at $x = 3$

g' changes from (−) to (+)

XXXXXXXXXXXXXXXXXXXX

Lesson 4: Concavity and the Second Derivative Test

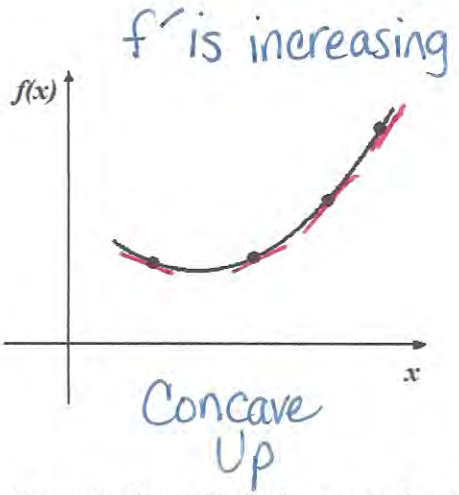
Topic 5.6: Determining Concavity of Functions over Their Domains

In our last lesson, we found intervals on which a function was increasing or decreasing and discovered how the first derivative describes the direction of a function (how the y-values are changing). In this lesson, we will see that the derivative of $f'(x)$, that is, the second derivative, will describe the concavity of the original function. Concavity tells us the direction of the curve, how it bends . . .

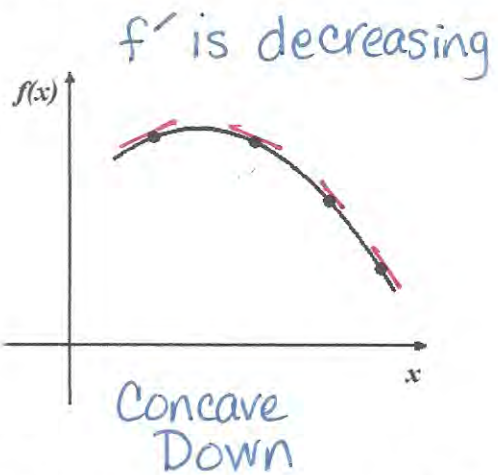
Just like direction, concavity of a curve can change, too. The points of change are called *inflection points*.

CONCAVITY EXPLORATION:

Draw small tangent lines at points along the curves below. What do you notice about the slopes of the tangent lines (the derivatives) as you move from left to right at these points?



A. ***Tangent lines lie below graph of f.***



B. ***Tangent lines lie above graph of f.***

CONCAVITY DEFINITION:

Let f be differentiable on an open interval I. The graph of f is:

- ❑ Concave upward on I if $f'(x)$ is increasing on the interval I; and,
- ❑ Concave downward on I if $f'(x)$ is decreasing on the interval I.

TEST FOR CONCAVITY

- ❑ If $f''(x) > 0$, then graph of f is concave up.
- ❑ If $f''(x) < 0$, then graph of f is concave down.

EX #1: Given $f(x) = \frac{1}{3}x^3 - x$ determine the open intervals on which the graph is concave upward or downward.

$f'(x) = x^2 - 1$

$f''(x) = 2x$

$2x = 0$

$\underline{\underline{x = 0}}$ P.O.I.?

f'' $- - -$ 0 $+ + +$

0 POI

$f'(x) = 0$

$x^2 - 1 = 0$

$x = \pm 1$

CRITICAL NUMBERS

$f(x)$ is concave up on $(0, \infty)$ because $f''(x) > 0$

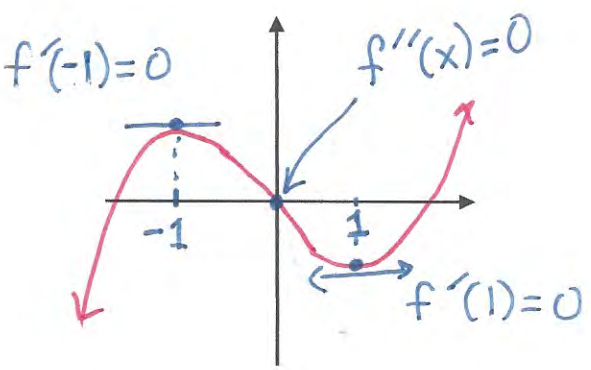
$f(x)$ is concave down on $(-\infty, 0)$ because $f''(x) < 0$

at $x = 0$ $f(x)$ has a point of inflection since $f''(0) = 0$ and $f''(x)$ changes from negative to positive here.

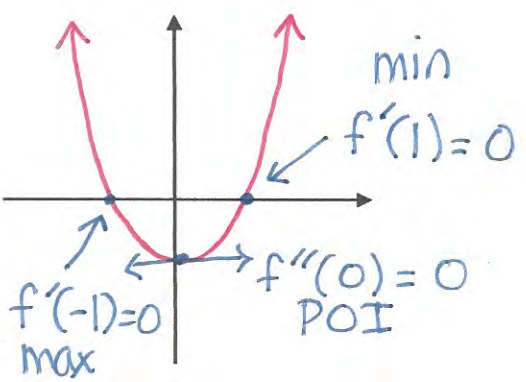
EX #2: Graphs and Derivatives

The concavity $(f''(x))$ and direction $(f'(x))$ of the function $(f(x))$ is related to the slope of the derivative.

$f(x) = \frac{1}{3}x^3 - x$



$f'(x) = x^2 - 1$



SUMMARY:

$f'(x)$ $+ +$ 0 $- - -$ 0 $+ +$

f ↗ inc −1 ↘ dec 1 ↗ inc

$f''(x)$ $- - -$ 0 $+ + +$

f ∩ dec 0 ∪ inc

f'

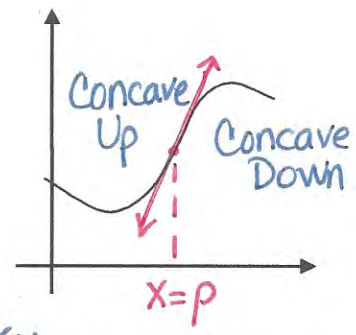
XXXXXXXXXXXXXXXXXXXX

DEFINITION POINT OF INFLECTION

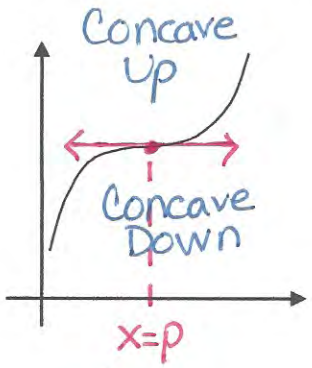
Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward or downward to upward at the point.

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

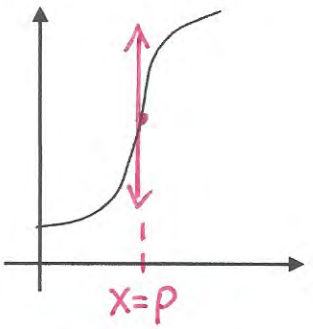
Tangent Lines and Points of Inflection:



$f''(p) > 0$ $f''(p) = 0$

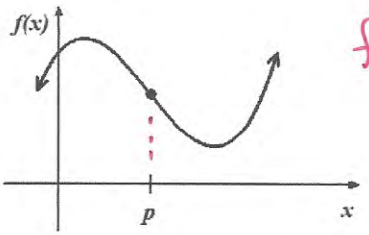
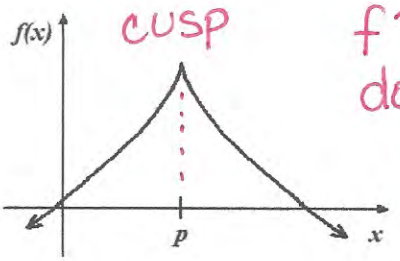
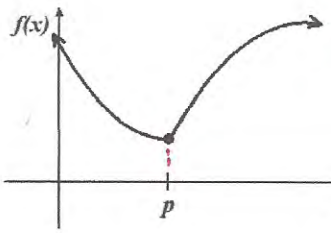
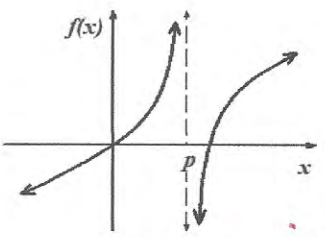


$f''(p) = 0$



$f''(p)$ dne

Let's explore the possibility that a graph could change concavity at a **discontinuity**, say a vertical tangent line; or, where f'' **doesn't exist**, such as at a cusp. In the case of a vertical tangent line, where the x-value is not in the domain of the function, then the value $x = p$ **WILL NOT BE AN INFLECTION POINT.** Here are a few examples:

Inflection Point at $x = p$	No Inflection Point at $x = p$
 $f''(p)=0$	 $f''(p)$ does not exist
 $f''(p)$ D.N.E.	 $x = p$ is V.A.

Some Important Thoughts:

We will use a **second derivative sign chart** to determine intervals of concavity, as well as, actual inflection points. The "possible points of inflection" can be called **critical values of $f'(x)$**.

Remember, **concavity can change at a discontinuity**, such as a vertical asymptote, but it won't be an actual inflection point. In order to classify an x-value as an inflection point of a function $f(x)$, the x-value must be **in the domain of $f(x)$**.

EX #3: Determine any points of inflection and discuss concavity of the graph of $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$4x^2(x-3)=0$

$x=0,\ x=3$

Test # -1 , 1 , 3

Critical # ∪ 0 ∩ 2 ∪

Sign $f''(x)$ +++0 - - - 0 +++

$f''(x) = 12x^2 - 24x$

$12x(x-2)=0$

$x=0,\ x=2$

1) Points of inflection occur at $f(0)=0$ and $f(2)=-16$

2) $f(x)$ is concave down on $(0,2)$ b/c $f''(x)<0$

3) $f(x)$ is concave up on $(-\infty,0)\cup(2,\infty)$ b/c $f''(x)>0$

EX #4: Find the inflection point(s) and the open interval(s) of concavity. Justify.
Given: $g(x) = \frac{1}{2}x + \sin x$ on $[0, 2\pi]$

$g'(x) = \frac{1}{2} + \cos x$

$g''(x) = -\sin x$

$-\sin x = 0$

$x = \pi$

$\pi/2$ $3\pi/2$

0 - - - π +++ 2π

$g(\pi) = \frac{1}{2}\pi + \sin\pi$

$g(\pi) = \frac{\pi}{2}$

1) inflection point is $(\pi, \pi/2)$ b/c $g''(\pi)$ changes signs from negative to positive at $x=\pi$

2) $g(x)$ is concave down on $(0,\pi)$ since $g''<0$

3) $g(x)$ is concave up on $(\pi, 2\pi)$ since $g''>0$.

EX #5: Use a calculator to find how many times the graph of $y = x^2 - 3 + \sin(2x - 1)$ changes concavity on the interval $[0, 2\pi]$.

Solve$(2 - 4\sin(2x-1) = 0, x \mid 0 < x < 2\pi)$

$x = \frac{12n\pi + \pi + 6}{12}$; $x = \frac{12n\pi + 5\pi + 6}{12}$

$n = 0, 1, 2, 3, \ldots$

$x = 0.7618$, $x = 1.8089$, $x = 3.9033$, $x = 4.9505$, $x \neq 7.0449$

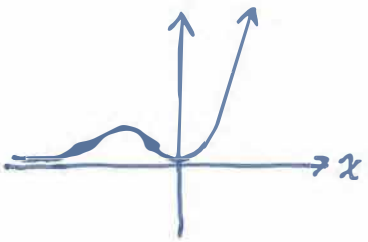
4 times

EX #6: Using a calculator, find the value(s) of x at which the graph of $y = e^x x^2$ changes concavity.

TI-NSPIRE CX-CAS

Press MENU: 6:5 (twice)

$\boxed{\begin{array}{l} x \approx -0.5857 \\ x \approx -3.4142 \end{array}}$



Topic 5.7: Using the Second Derivative Test to Determine Extrema

Using the Second Derivative to find Extrema of a Function

Similarly, the *SECOND DERIVATIVE* is the *FIRST DERIVATIVE* of $f'(x)$, so the same relationships that exist between $f(x)$ and $f'(x)$ must also exist between $f'(x)$ and $f''(x)$. We can use the SECOND DERIVATIVE TEST as an alternate method for locating relative extrema.

If $x = c$ is a critical number of $f(x)$, then $f'(c) = 0$ or $f'(c)$ does not exist. This makes $x = c$ a "possible" maximum or minimum value of the function $f(x)$.

$f''(x)$	$f(x)$	$f'(x)$
positive	concave up	increasing
negative	concave down	decreasing
neither	possible point of inflection here	possible relative max/min here

EX #7: Use the second derivative test to find the relative extrema for each function. Justify.

A. $g(x) = x^4 - 8x^3 - 5$

$g'(x) = 4x^3 - 24x^2$

$4x^3 - 24x^2 = 0$

$4x^2(x-6) = 0$

$x = 0, \quad x = 6$ critical numbers

$g''(x) = 12x^2 - 48x$

$g''(0) = 12(0)^2 - 48(0)$

$g''(0) = 0$

$g''(6) = 12(6)^2 - 48(6)$

$g''(6) = 144$

$x = c$	$f(c)$	$f''(c)$
0	-5	zero
6	-437	positive

$g(0) = -5$; $g(6) = -437$

Conclusions:

1) Local min occurs at $x=6$ and is -437 b/c $g''(6)>0$ g is concave up.

2) No conclusion above $x=0$ since $g''(0)=0$.

B. $f(x) = \frac{1}{2}x - \sin x$ on $-2\pi \le x \le \pi$

$f'(x) = \frac{1}{2} - \cos x$

$\frac{1}{2} - \cos x = 0$

$\cos x = \frac{1}{2}$

$x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$ critical values

$f''(x) = \sin x$

$f(-\frac{5\pi}{3}) = -\frac{5\pi}{6} - \sin(-\frac{5\pi}{3})$

$f(-\frac{\pi}{3}) = -\frac{\pi}{6} - \sin(-\frac{\pi}{3})$

$f(\pi/3) = \frac{\pi}{6} - \sin(\frac{\pi}{3})$

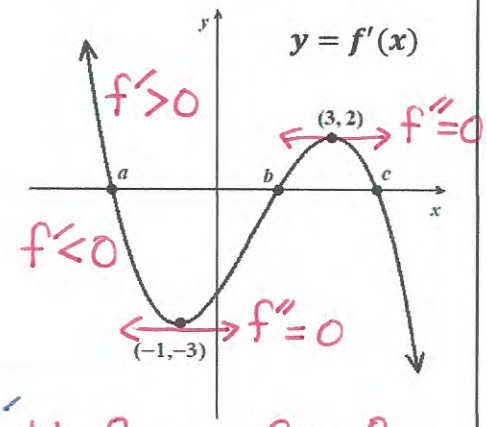
$x = c$	$f(c)$	$f''(c)$
$-\frac{5\pi}{3}$	$-\frac{5\pi}{6} - \frac{\sqrt{3}}{2}$	positive
$-\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} - \frac{\pi}{6}$	positive
$\frac{\pi}{3}$	$\frac{\pi}{6} - \frac{\sqrt{3}}{2}$	negative

Conclusions:

1) Local mins are ≈ -3.484 @ $x = -5\pi/3$ and ≈ -0.342 @ $x = \pi/3$ since $f''(x) > 0$ and f is concave up.

2) Local max value is about 0.342 @ $x = -\pi/3$ since $f''(x) < 0$ and f is concave down.

EX #8: The graph of f', the derivative of a function f is shown.



A. On what open interval(s) is f increasing? Justify.

f increases where $f'(x) > 0$
this occurs on $(-\infty, a)$
and (b, c)

f' ++ 0 - - - - 0 ++ 0 - -
f ↗ a ↘ b ↗ c ↘

B. At what value(s) of x does f have a local maximum or minimum? Justify.

f'' - - - 0 +++ 0 - - -
f ∩ -1 ∪ 3 ∩

f has a local max at $x = a$ and $x = c$ because $f'(x)$ changes sign from positive to negative.

f has a local min at $x = b$ because $f'(x)$ changes sign from negative to positive

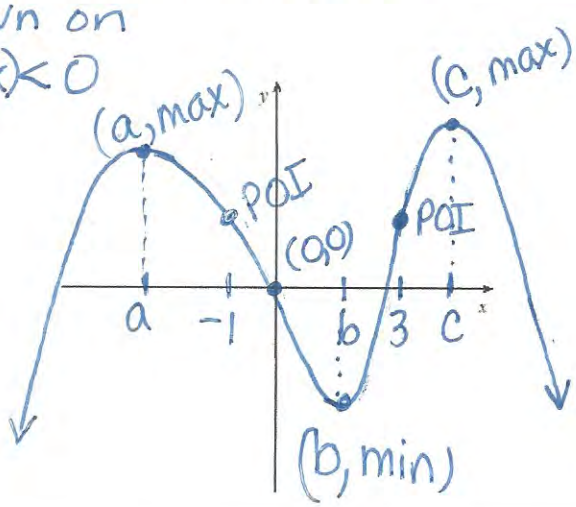
C. State the inflection points of f. Justify.

f has points of inflection at $x = -1$ and $x = 3$ because $f''(x) = 0$ here.

D. On what interval(s) is the function f concave up? concave down?

f is concave up on $(-1, 3)$ since $f''(x) > 0$ and f is concave down on $(-\infty, -1) \cup (3, \infty)$ since $f''(x) < 0$

E. Assume that $f(0) = 0$; can you sketch a graph of $f(x)$?



EX #9: In the table at right, selected values for $g(x)$, $g'(x)$ and $g''(x)$ are shown. The function $y = g(x)$ is a twice-differentiable function and continuous on $-4 \le x \le 5$.

x	$g(x)$	$g'(x)$	$g''(x)$
-4	0	-6	-1
3	4	0	-2
5	8	3	0

A. Explain why there must be a value $x = c$ on $(-4, 5)$ such that $g(c) = 7$.

$g(-4)=0$ $g(5)=8$; Since $0<7<8$ and f is continuous on $[-4,5]$, by IVT there must be a value $x=c$ on $(-4,5)$ where $g(c)=7$.

B. Explain why there must be a value $x = d$ on $(-4, 5)$ such that $g'(d) = \frac{8}{9}$

$g'(d) = \frac{g(5)-g(4)}{5-(-4)} = \frac{8}{9}$; since f is continuous on $[-4,5]$ and differentiable on $(-4,5)$; by MVT there is a value such that $g'(d) = \frac{8}{9}$.

C. Does $g(x)$ have a local maximum, local minimum or neither at $x = 3$? Justify.

$g'(3)=0$ critical number and $g''(3)=-2$. So $g(x)$ is concave down and $g(3)$ is a local maximum of 4 at $x=3$ by 2nd Derivative Test for Extrema

D. Can concavity of $g(x)$ be determined on the interval $-4 < x < 5$? Why or why not?

No, there is not enough information to determine concavity on $-4<x<5$. We only have endpoints and $x=3$ so infinitely many points are missing to establish concavity

SUMMARY OF FIRST AND SECOND DERIVATIVE TESTS RELATING FUNCTION BEHAVIOR:

	$f'(x) > 0$	$f'(x) < 0$	$f'(x) = 0$
$f''(x) > 0$	Increasing and Concave Up	Decreasing and Concave Up	Relative Minimum and Concave Up
$f''(x) < 0$	Increasing and Concave Down	Decreasing and Concave Down	Relative Maximum and Concave Down
$f''(x) = 0$	Increasing and Inflection Point	Decreasing and Inflection Point	Function is "smooth, level" and a possible inflection point

Lesson 5: Curve Sketching

Topic 5.8: Sketching Graphs of Functions and Their Derivatives

Curve Sketching is a procedure for analyzing a function and its behavior without the aid of a graphing utility. We can use the relationships of f, f', and f'' over specific intervals to quickly sketch the graph of the function $y = f(x)$. We already know how to find domain, range, intercepts and asymptotes. In the last two lessons we learned how the first and second derivatives can be used to determine four basic behaviors and curvature of a function. If we know the sign of f' and f'' over some given interval, we know exactly what the graph of $f(x)$ looks like on that interval.

EX #1: Complete the table below by sketching the general shape of the curve when each condition is met.

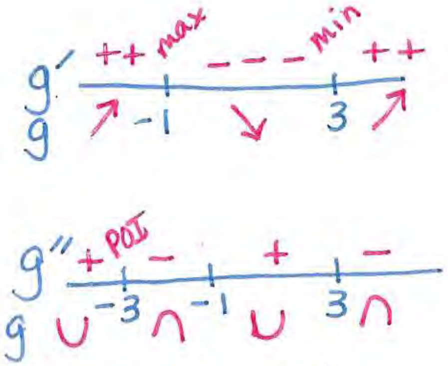
	$f'(x) > 0$ increasing	$f'(x) < 0$ decreasing
$f''(x) > 0$ concave up		
$f''(x) < 0$ concave down		

EX #2: Sketch a possible graph of the function $y = g(x)$ that satisfies the following conditions:

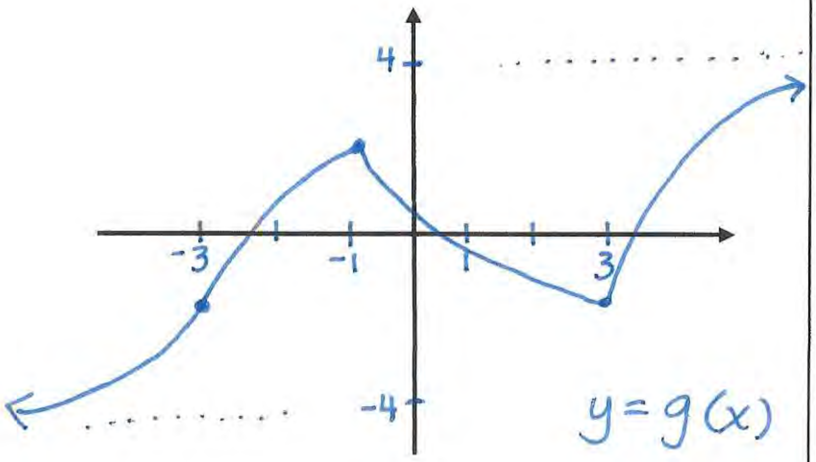
i. $g'(x) > 0$ on $(-\infty, -1) \cup (3, \infty)$ and $g'(x) < 0$ on $(-1, 3)$

ii. $g''(x) > 0$ on $(-\infty, -3) \cup (-1, 3)$ and $g''(x) < 0$ on $(-3, -1) \cup (3, \infty)$

iii. $\lim_{x\to-\infty} g(x) = -4$ and $\lim_{x\to\infty} g(x) = 4$



y = -4 y = 4



Sketching Function Graphs from the Derivative Graph

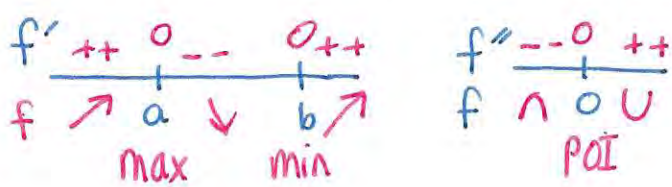
The Function Behavior Chart from our previous lesson can be a visual aid to using proper vocabulary as well as helping us make graphical representations from verbal descriptions.

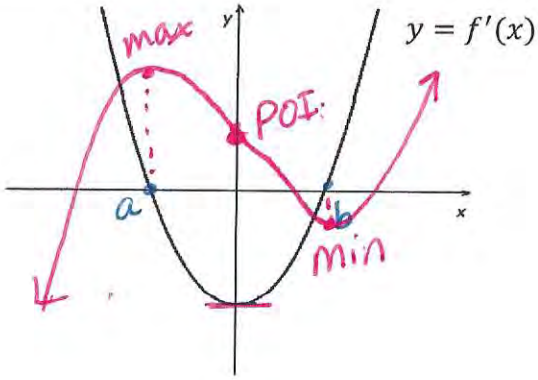
Function Behavior				
$f(x)$	$\cup$	$\cap$	$\nearrow$	$\searrow$
$f'(x)$	$\nearrow$	$\searrow$	$+$	$-$
$f''(x)$	$+$	$-$		

Symbol	Vocabulary
$+$	Positive
$-$	Negative
$\nearrow$	Increasing
$\searrow$	Decreasing
$\cup$	Concave Up
$\cap$	Concave Down

Discovering Key Concepts of Analyzing f' Graphs

EX #3: Can you sketch a possible graph of f given f'?

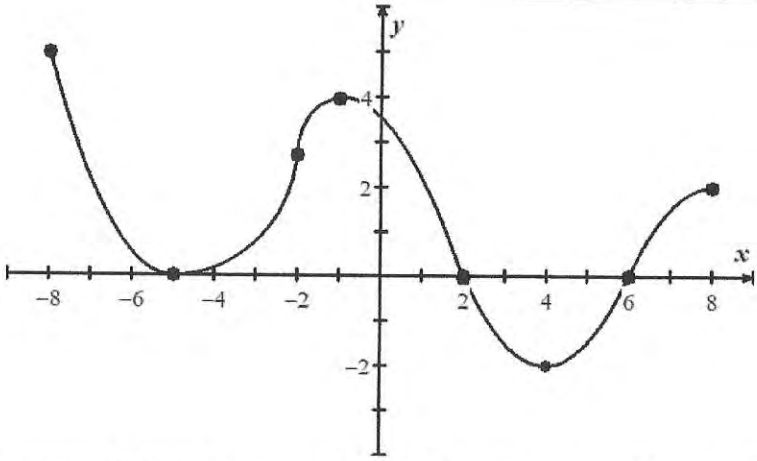




1. Zeros of $f'(x)$ may be extrema of $f(x)$.
2. Extrema of $f'(x)$ are possible inflection points for $f(x)$.
3. When $f'(x)$ is above the x-axis then $f(x)$ is increasing.
4. When $f'(x)$ is below the x-axis then $f(x)$ is decreasing.
5. If $f'(x)$ is increasing then $f''(x) > 0$, so $f(x)$ is concave up.
6. If $f'(x)$ is decreasing then $f''(x) < 0$, so $f(x)$ is concave down.

Topic 5.9: Connecting a Function, Its First Derivative, and Its Second Derivative

EX #4:



A. Suppose the graph shown above is the graph of a function $y = f(x)$ on the interval $[-8, 8]$:

i. On what open interval(s) is $f(x)$ both increasing and concave up?

$(-5,-2) \cup (4,6)$

ii. On what open interval(s) is $f(x)$ both decreasing and concave down?

$(-1,2)$

iii. At what x-value(s) does $f(x)$ have inflection points?

$x=-2,\ x=2,\ x=6$

B. If the graph above is the graph of the derivative $y = f'(x)$ on $[-8, 8]$:

i. On what open interval(s) is $f(x)$ decreasing ? Justify.

f decreases on $(2,6)$ where $f'(x)<0$

ii. At what x-value(s) does $f(x)$ have a local maximum ? Justify.

At $x=2$ f has a local max. b/c $f'(x)$ changes sign from positive to negative.

iii. On what open interval(s) is $f(x)$ concave down? Justify.

f is concave down where $f'(x)$ decreases on $(-8,-5) \cup (-1,4)$

iv. At what x-value(s) does $f(x)$ have an inflection point? Justify.

At $x=-5$, $x=-1$, and $x=4$ b/c $f''(x)=0$ and f'' changes sign from $(-)$ to $(+)$ or $(+)$ to $(-)$

C. Suppose the graph shown above is the graph of a function $y = f''(x)$ on the interval $[-8, 8]$:

i. On what open interval(s) is $f(x)$ concave down? Justify

Where $f''(x)<0$, so $f(x)$ is concave down on $(2,6)$

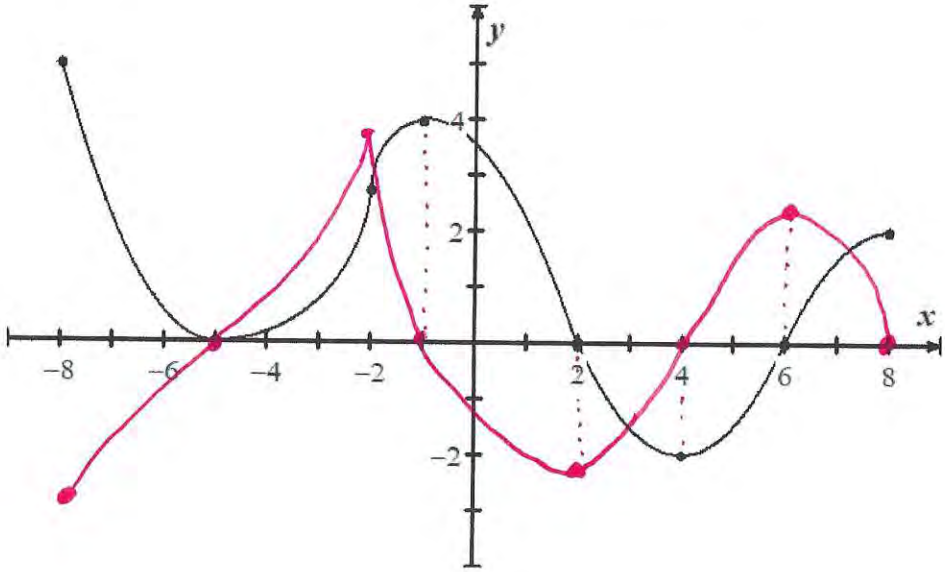
ii. At what x-value(s) does $f(x)$ have an inflection point? Justify.

$x=2$ and $x=6$ where $f''(x)=0$ and f'' changes signs from $(+)$ to $(-)$ or $(-)$ to $(+)$

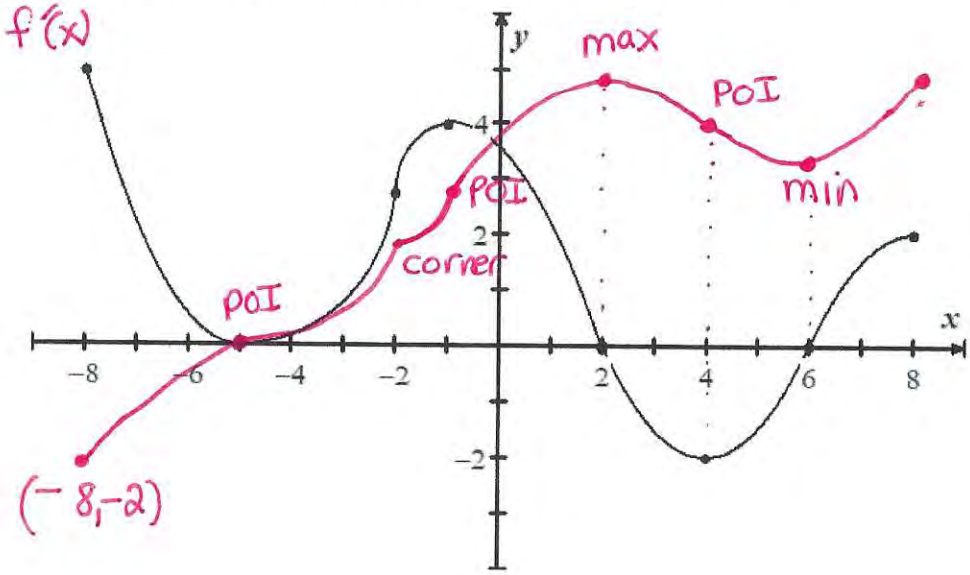
XXXXXXXXXXXXXXXXXXXX

EX #4 (continued)

D. Suppose the graph below is the graph of a function $y = f(x)$ on the interval $[-8, 8]$. On the same set of axes, sketch and label a possible graph of $f'(x)$ in red.



E. If the graph below is the graph of the derivative $y = f'(x)$ on $[-8, 8]$. Sketch and label a possible graph of $y = f(x)$ given that $f(-8) = -2$.



Guidelines for Analyzing the Graph of a Function:

1. Determine the domain, intercepts, asymptotes, symmetry of the graph.
2. Find critical points and intervals where the function is increasing and decreasing.
3. Determine local maximum and minimum points.
4. Determine concavity and find points of inflection.
5. Sketch the curve.

EX #5: Analyze and sketch the graph of $g(x) = (x+2)(x-1)^2$

$g(x) = x^3 - 3x + 2$

A. Find the x and y-intercepts.

x-int: $(-2,0)$ $(1,0)$ y-int: $(0,2)$

B. Find the first and second derivatives.

$g'(x) = 3x^2 - 3$

$3(x^2-1) = 0$

$x = -1,\ x = 1$

$g''(x) = 6x$

$6x = 0$

$x = 0$

C. Complete a sign chart for $g'(x)$ to find intervals where $g(x)$ is increasing or decreasing.

g': ++ 0 -- 0 ++

g: ↗ -1 ↘ 1 ↗

$g(x)$ increases on $(-\infty,-1) \cup (1,\infty)$

$g(x)$ decreases on $(-1,1)$

g'': --- 0 +++

g: ∩ ∪

D. Determine any relative extrema.

rel. max: 4 at $x = -1$

rel. min: 0 at $x = 1$

E. Complete a sign chart for $g''(x)$ to find intervals where $g(x)$ is concave up or concave down.

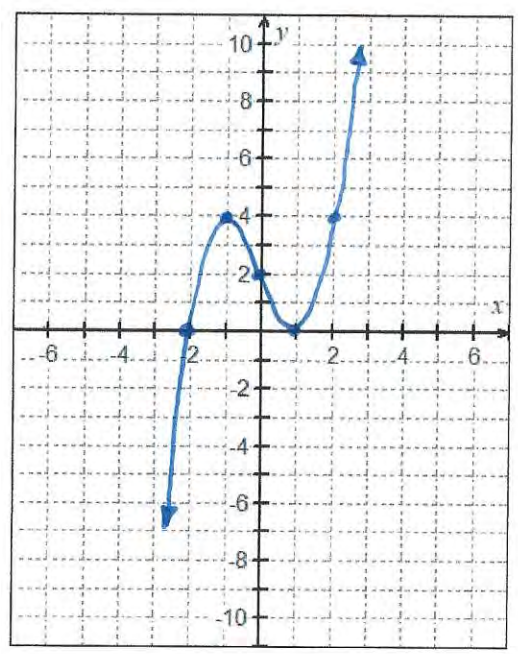
g is concave down $(-\infty,0)$

g is concave up $(0,\infty)$

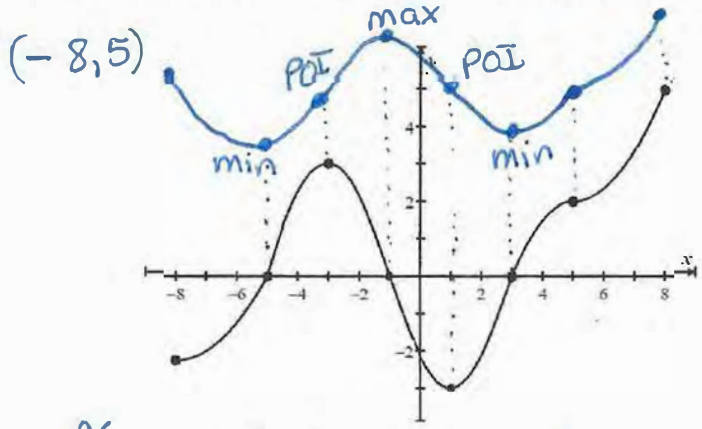
F. Identify any points of inflection.

$(0,2)$ is P.O.I.

G. Sketch the graph of $g(x)$.



EX #6: The graph of $y = f'(x)$ is shown below, on the interval $[-8, 8]$. If $f(-8) = 5$. On the same set of axes, sketch a possible graph of the function $y = f(x)$.



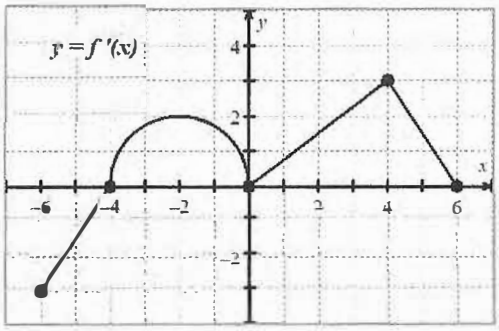
f': − 0 + − +

f: −8 ↘ −5 ↗ −1 ↘ 3 ↗ 8

f'': + + + + 0 − − − 0 + + + 0 + + +

f: −8 ∪ −3 ∩ 1 ∪ 5 ∪ 8

EX #7: The function $f(x)$ is defined and differentiable on the closed interval $[-6, 6]$. The graph of $y = f'(x)$, the derivative of f, consists of three line segments and a semicircle, shown in the figure at right. For each question below find the values for $y = f(x)$ on the interval $-6 < x < 6$ and justify your reason.



A. Find the x-coordinate of each critical point for $y = f(x)$.

$f'(x) = 0$; $x = -4$ and $x = 0$

B. Find the x-coordinate of each relative extrema of $y = f(x)$ and label as a maximum or minimum.

At $x = -4$, f' changes sign from negative to positive, so $x = -4$ is a relative minimum.

C. Find the open interval(s) over which the function $y = f(x)$ is increasing or decreasing.

Where $f'(x) > 0$, f is increasing on $(-4, 0) \cup (0, 6)$

Where $f'(x) < 0$, f is decreasing on $(-6, -4)$

D. Find the x-coordinate of each point of inflection for $y = f(x)$.

At $x = 4$ & $x = -2$, $f''(x)$ changes sign from positive to negative.

At $x = 0$, $f''(x)$ changes sign from negative to positive

$f''(-2) = 0$ and $f''(4)$ dne & $f''(0)$ dne

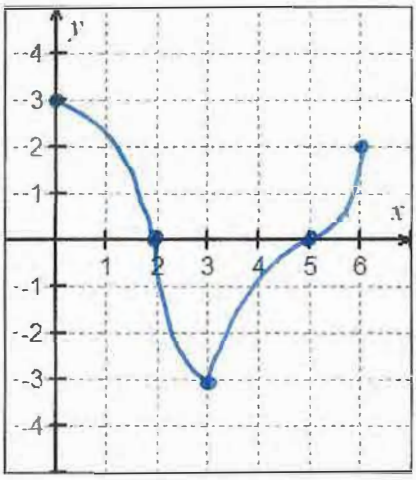
E. Find the interval(s) over which the function $y = f(x)$ is concave up or concave down.

$f''(x) > 0$, f is concave up $(-6, -2) \cup (0, 4)$

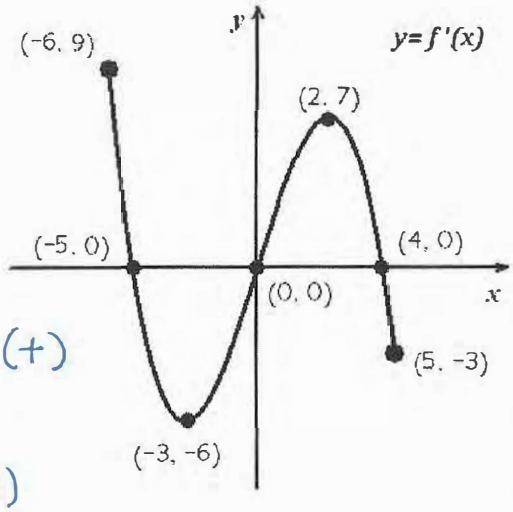
$f''(x) < 0$, f is concave down $(-2, 0) \cup (4, 6)$

EX #8: **Let $h(x)$ be a function that is continuous on the interval $[0, 6]$. The function h is twice differentiable except at $x = 3$, where the derivative of h does not exist. The function and its derivatives have the properties indicated in the table below. Use the table to sketch a graph of $y = h(x)$.**

x	0	$0 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$	5	$5 < x < 6$
$h(x)$	3	Positive	0	Negative	-3	Negative	0	Positive
$h'(x)$	-3	Negative	-3	Negative	DNE	Positive	1	Positive
$h''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive



EX #9: The function f is defined and differentiable on the closed interval $[-6, 5]$. The graph of $y = f'(x)$ the derivative of f, is shown in the figure below and has horizontal tangent lines at $x = -3$ and $x = 2$.



A. Find the x-coordinate(s) of each relative extrema on $-6 < x < 5$. Identify as a max or min. Justify.

max: $x = -5$, $x = 4$; f' changes sign from (+) to (−)

min: $x = 0$; f' changes from (−) to (+)

B. Find the x-coordinate of each point of inflection for $y = f(x)$. Justify.

f'' changes sign from (−) to (+) at $x = -3$ and from (+) to (−) at $x = 2$.

C. Let $g(x) = x^3 - f(x)$. Find $g'(-3)$.

$g'(x) = 3x^2 - f'(x)$

$g'(-3) = 3(-3)^2 - f'(-3)$

$g'(-3) = 27 - (-6)$

$g'(-3) = 33$

Lesson 6: Optimization

Topic 5.10: Introduction to Optimization Problems
Topic 5.11: Solving Optimization Problems

Optimization is one of the most common applications in calculus. We often hear terms like least time, greatest profit, shortest distance, maximum volume, minimum waste. In order to optimize problems of this sort, we first need to find the quantity that needs to be minimized or maximized. Students often find this to be a challenging lesson to master due to the difficulty presented with finding the quantity to optimize, the constraints, and then writing equations for each part of the problem. ***Just be sure to read the problem carefully first. Then identify the quantity to be optimized and the constraint. If a quantity is stated to be a fixed value, this will be your constraint, which must be true regardless of your solution.***

EX. #1: Find two positive numbers such that the sum of the first and twice the second is 64 and whose product is a maximum.

$x =$ 1ST number
$y =$ 2nd number

$x + 2y = 64$

$y = \frac{64-x}{2}$

$xy \Rightarrow$ maximum

$f(x) = x\left(\frac{64-x}{2}\right)$

$f(x) = \frac{64x - x^2}{2}$

$f(x) = 32x - \frac{1}{2}x^2$

$f'(x) = 32 - x$

$x = 32$

$x = 32$

$y = \frac{64-32}{2} = 16$

The numbers are 32 and 16

Optimization Problems Over Closed, Bounded Intervals

EX #2: A rancher plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

$P = x + 2y$

$A = xy$

$180{,}000 = xy$

$y = \frac{180{,}000}{x}$

$P(x) = x + 2\left(\frac{180{,}000}{x}\right)$

$P(x) = x + \frac{360{,}000}{x}$

$P'(x) = 1 - \frac{360{,}000}{x^2}$

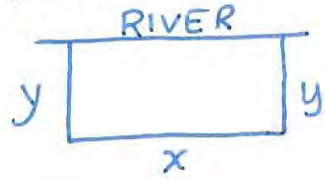
$1 - \frac{360{,}000}{x^2} = 0$

$1 = \frac{360{,}000}{x^2}$

$x = \pm\sqrt{360{,}000}$

$x = 600\text{ m}$

$y = 300\text{ m}$



domain $x > 0$

P': $- -$ 0 $+ +$ (600)

The pasture measures 300m by 600m

EX #3: An open box is being constructed from a piece of sheet metal 18 inches by 30 inches by cutting out squares of equal size from the corners and bending up the sides. What size squares should be cut to make a box of maximum volume? What is the volume?

① $V(x) = x(18-2x)(30-2x)$

$V(x) = 4x^3 - 96x^2 + 540x$

② $V'(x) = 12x^2 - 192x + 540$

$12(x^2 - 16x + 45) = 0$

$(x^2 - 16x + 64) + 45 - 64 = 0$

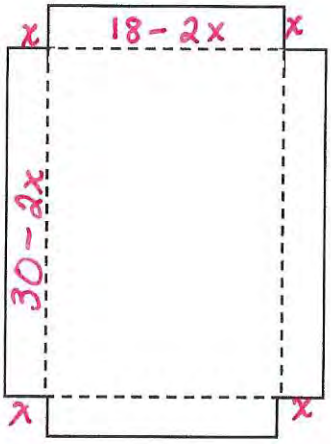
$(x-8)^2 - 19 = 0$

$(x-8)^2 = 19$

$x = 8 \pm \sqrt{19}$

$x = 8 - \sqrt{19}$ only

$x \approx 3.641$ in.



domain: $0 < x < 9$

③ $V(8-\sqrt{19}) \approx 886.552 \text{ in}^3$

④ V' ++ max - - ↗ 3.641 ↘

⑤ The volume of approx 886.552 in^3 will be maximized by cutting squares from the corners of sheet metal that are $x = 8 - \sqrt{19}$ inches because $V'(x) = 0$ at $x \approx 3.641$ in. and signs change from positive to negative here.

Strategy for Solving Optimization Problems

1. ***Read*** the problem carefully. ***Underline*** key information.
2. ***Draw*** a picture. ***Label*** constants and variables for unknowns.
3. Write a ***primary equation,*** an equation that gives a formula for the quantity to be optimized.
4. Identify the constraint (limiting factor) and write a ***secondary equation***, and equation that provides information needed to complete the primary equation. ***This equation will determine domain constraints.***
5. Solve the secondary equation for the variable to be optimized, plug it into the primary equation to create an equation in terms of a single variable.
6. Simplify the primary equation and ***determine a feasible domain***.
7. ***Differentiate*** and find ***critical values*** within your domain.
8. Determine the ***Absolute Extrema*** and ***JUSTIFY***.
9. Answer the question in a complete sentence using the appropriate units.

Optimization Problems Over an Unbounded Interval

In the last few examples we were guaranteed that the functions had absolute extrema by the Extreme Value Theorem. Let's explore a few situations for which the domain is neither closed nor bounded.

EX. #4: What are the dimensions of an aluminum can that can hold 40 in^3 of soda and that uses the least amount of aluminum? Assume the can is cylindrical and capped on both ends.

① $SA = 2\pi r^2 + 2\pi r h$

$V = 40\ in^3$

$\pi r^2 h = 40$

$h = \frac{40}{\pi r^2}$

② $S(r) = 2\pi r^2 + 2\pi r\left(\frac{40}{\pi r^2}\right)$

$S(r) = 2\pi r^2 + \frac{80}{r}$

③ $S'(r) = 4\pi r - \frac{80}{r^2}$

④ $4\pi r = \frac{80}{r^2}$

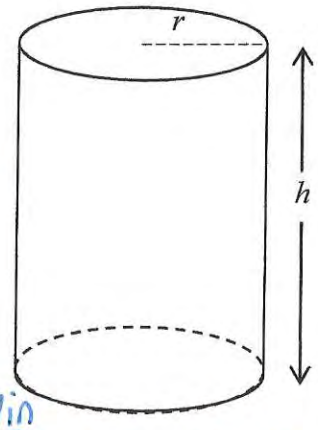
$r^3 = \frac{80}{4\pi}$

$r = \sqrt[3]{\frac{20}{\pi}} \approx 1.853\ in.$

⑤ S': $- -$ min $+ +$; $0 \searrow 1.853 \nearrow$

⑥ $h = \frac{40}{\pi\left(\sqrt[3]{20/\pi}\right)} \approx 3.707\ in$

domain $SA > 0$, $V = 40\ in^3$



⑦ The can measures $r \approx 1.853\ in.$ $h \approx 3.707\ in.$ with Volume $40\ in^3$ of soda since $S'(1.853) = 0$ and $S'(r)$ changes sign from negative to positive here.

EX. #5: Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

① By distance formula

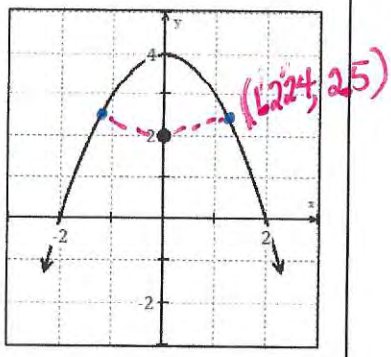
$d(x) = \sqrt{(x-0)^2 + (4 - x^2 - 2)^2}$

$d(x) = \sqrt{x^2 + (2 - x^2)^2}$

$d(x) = \sqrt{x^2 + 4 - 4x^2 + x^4}$

$*d(x) = \sqrt{x^4 - 3x^2 + 4}$

* radicand is sufficient for finding the smallest critical value



② $g(x) = x^4 - 3x^2 + 4$

$g'(x) = 4x^3 - 6x$

$2x(2x^2 - 3) = 0$

$x = 0 \quad x = \pm\sqrt{\frac{3}{2}}$

③ g': $-$ $+$ $-$ $+$; $\searrow -\sqrt{\frac{3}{2}} \nearrow 0 \searrow \sqrt{\frac{3}{2}} \nearrow$; min, max, min

④ $f\left(\pm\sqrt{\frac{3}{2}}\right) = 4 - \left(\pm\sqrt{\frac{3}{2}}\right)^2 = \frac{5}{2}$

⑤ By 2nd Derivative Test $g''(x) > 0$ for all x, so the closest points to $(0,2)$ on the graph of $y = 4 - x^2$ are $\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ and $\left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$.

Methods and Conditions for Justification

Method #1: Use Extreme Value Theorem when given endpoints. *This method requires finite endpoints and a continuous function over the relevant interval.*

Method #2: Modified version of 1st Derivative Test for Relative Extrema. *Use this method when you don't have both endpoints at a closed interval, but you have a half-open interval or open interval. This method requires a continuous function.*

Method #3: Modified version of 2nd Derivative Test for Relative Extrema. *Use this method when the 2nd derivative is easy to obtain.*

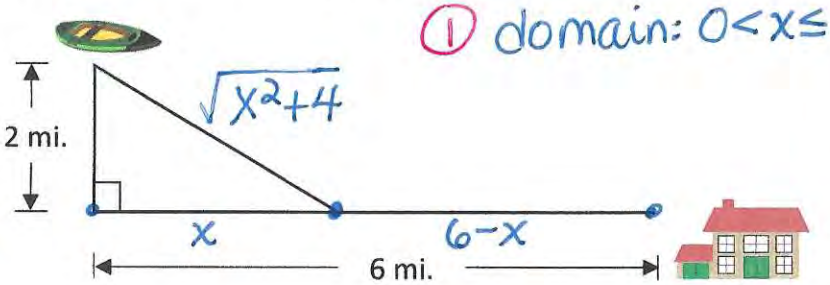
NOTE: ***It is important to show the test works FOR ALL VALUES IN THE RELEVANT DOMAIN, to guarantee a GLOBAL argument rather than a LOCAL extrema.***

EX #6: The training facility for a sport club is situated along a straight shoreline. You row your kayak out 2 miles from the nearest point on the shoreline and then row 6 miles down the beach, each day to train for an event that includes rowing and running. If you can row at a rate of 3 miles per hour and jog at a rate of 5 miles per hour, what is the least amount of time required to reach the clubhouse. How far from the clubhouse should you beach your kayak?

① domain: $0 < x \le 6$

② row: 3 mph
jog: 5 mph

$D = RT$



③ T = Row time + Jog time

$$T(x) = \frac{\sqrt{x^2+4}}{3} + \frac{6-x}{5}$$

④ $$T'(x) = \frac{2x}{6\sqrt{x^2+4}} - \frac{1}{5}$$

⑤ $T'(x) = 0$

$$\frac{x}{3\sqrt{4+x^2}} = \frac{1}{5}$$

$$5x = 3\sqrt{x^2+4}$$

$$25x^2 = 9(x^2+4)$$

$$16x^2 = 36$$

$$x^2 = \frac{9}{4}$$

$$x = \pm\frac{3}{2} = 1.5 \text{ mi.}$$

$$T(1.5) = \frac{26}{15} \text{ hrs}$$

⑥ T': $- -$ on $(0, \frac{3}{2})$, $+ +$ on $(\frac{3}{2}, 6)$

$0 \searrow \frac{3}{2} \nearrow 6$

⑦ The least amount of time is $T(\frac{3}{2}) \approx 1.733$ hrs.
if you beach the kayak at 1.5 miles since $T'(\frac{3}{2})$ changes signs from negative to positive.

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EX #7: Justin Case found a four-foot piece of wire left on a construction site. He is learning about optimization in Calculus and wants to cut the wire to create a circle and a square. How much of the wire should be used for each shape in order to maximize the total area?

Total Area = $A_{\circ} + A_{\square}$

$A = \pi r^2 + s^2$

○ $A = \pi r^2$
$C = 2\pi r$

□ $A = s^2$
$P = 4s$

① With 4 ft. of wire

4 = Circle circumference + Square perimeter

② D: $0 < s < 1$

$A_{\circ}$ is $r = \frac{2}{\pi}$

$* A\left(\frac{2}{\pi}\right) \approx 1.273$

③ $4 = 2\pi r + 4s$

$\frac{4-4s}{2\pi} = r$

Solve for "r"

$r = \frac{2-2s}{\pi}$

substitute

④ $A = \pi r^2 + s^2$

$$A(s) = \pi\left(\frac{2-2s}{\pi}\right)^2 + s^2$$

$$A(s) = \cancel{\pi}\left[\frac{4-8s+4s^2}{\pi^{\cancel{2}}}\right] + s^2\frac{\pi}{\pi}$$

$$A(s) = \frac{1}{\pi}\left[4-8s+4s^2+\pi s^2\right]$$

$$= \frac{1}{\pi}\left[(4+\pi)s^2 - 8s + 4\right]$$

$$A'(s) = \frac{1}{\pi}\left[2s(4+\pi) - 8\right] = 0$$

$$2s(4+\pi) = 8$$

$$(4+\pi)s = 4$$

$$s = \frac{4}{4+\pi} \approx 0.560$$

A' -- | ++
↘ .560 ↗
min

$*A(0) = 1.273$

$A(0.560) = 0.560$

$A(1) = 1$

If length of square is $s=0$, Area is maximized Therefore, the total area will be maximized if Justin uses all the wire to create a circle. This is an endpoint maximum.