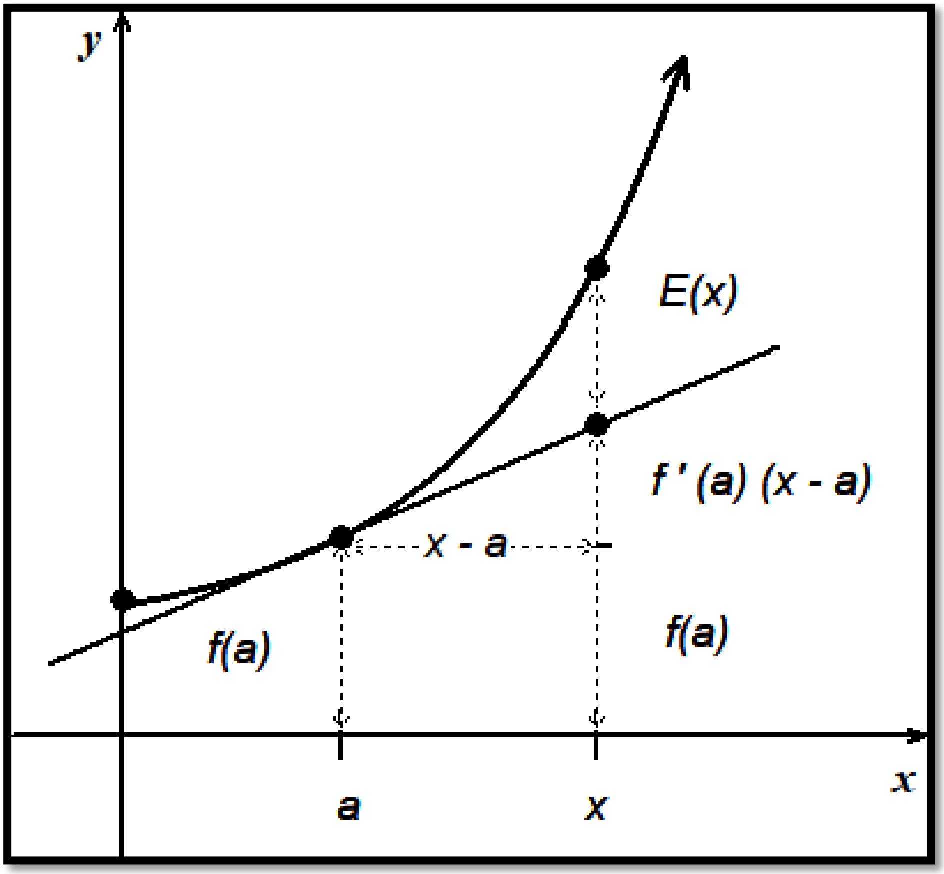


Name ______________________________

UNIT 4:

CONTEXTUAL APPLICATIONS OF DIFFERENTIATION NOTES

Lesson 1: Interpretation of the Derivative In Context

Topic 4.1: Interpreting the Meaning of the Derivative in Context

In our last two units we have defined the derivative graphically, algebraically, and numerically as a slope of the tangent line and the instantaneous rate of change at a point. In this lesson, we want to interpret the meaning of the derivative in a verbal manner, as it relates to the context of real-world problems.

A Look at Some Mathematical History

When Newton was developing his ideas of the infinitesimal calculus, namely when given a **fluent** (*we call function*) to find its **fluxion** (*we call derivative*), his thinking was concentrated on velocity. But, today, we know the derivative's importance reaches into so many other areas involving rates of change.

Let's look at the Leibniz notation one more time in a different way. He stated that the variable y depended on x, and if $y = f(x)$, then the derivative could be expressed as dy/dx. If we allow ourselves to think of "d" as representing *"a small difference in some process,"* we are reminded that the derivative is simply the limit of ratios in the familiar form

$$\frac{dy}{dx} = f'(x) = \frac{\Delta \; in \; y\text{−}values}{\Delta \; in \; x\text{−}values}.$$

Similarly, we can read $\frac{d}{dx}$ as a single symbol meaning "*the derivative with respect to x of ….*" which brings us to this new place of understanding the notation $\frac{d}{dx}(y)$ or $\frac{d}{dx}[f(x)]$, *"the derivative with respect to x of* y or $(f(x))$.

The Leibniz notation can be a little cumbersome when you need to specify the exact x-value where you need to evaluate the derivative. However, the benefit is that you can read so much about the rate of change through this expression. It's very easy to write $f'(3)$ but with Leibniz's notation we will write the derivative format like this:

$$\left.\frac{dy}{dx}\right|_{x=3}$$

EX #1: Sean is cleaning out his saltwater fish tank and needs to drain some of the water in order to adjust the saline levels. If the volume of the tank V, is in gallons, is being drained over time, t, in minutes. What is a meaningful interpretation of the following statement as it relates to Sean's fish tank?

$$\left.\frac{dV}{dt}\right|_{t=3} = -2$$

Using Meaningful Variables

Just like any other language course, we need to use proper names, spellings, and grammar conventions. AP Calculus is no different than AP Language. There are many real-world functions that depend on variables other than x, y, or time. Learning to read and write the correct notation will help you succeed on Free Response prompts throughout our course.

EX #2: Read each scenario and write the proper notation for the derivative.

A. The Alberta tar sands are one of the biggest oil reserves in the world. Yet extracting the fossil fuel costs more than the profits made. If the cost C (in dollars) to extract a barrel of oil is losing \$3 on every barrel, write the derivative.

B. The Deepwater Horizon oil spill is regarded as one of the largest environmental disasters in American history. The oil leak was discovered on April 22, 2010. BP originally estimated that the flow rate was about 790 cubic meters per day at the end of the first week.

C. The population P of the endangered mountain gorilla has an ongoing study tracking the number of gorillas since 1992. The population increased by 114 gorillas during 2003 of this same study.

EX #3: Suppose a planet is discovered in 2024; and, is named *Newtonia*. If the alien population $P(t)$ is recorded in millions of aliens, and t represents the number of years since the planet's discovery. Explain the meaning of each of the statements below.

A. $P'(3) = 0.5$

B. $P^{-1}(7.5) = 5$

C. $[P^{-1}(7.5)]' = 0.75$

Using Units to Interpret the Derivative

In the next examples, we will show how clues from the units can help you recognize interpretations of the derivative as they relate to real-world scenarios.

EX #4: Suppose you launch a water balloon from a giant slingshot. Let $s(t)$ give the distance, in feet, that the water balloon traveled from its initial starting point as a function of time, t, in seconds. Explain the meaning of the derivative notation.

$$\left.\frac{ds}{dt}\right|_{t=2} = s'(2) = 38\frac{ft.}{sec.}$$

EX #5: The total cost C ,in dollars, of an airplane flight is a function of the number of passengers, n can be calculated by Derivative Airlines using the function $C = f(n)$. What does it mean to say $f'(182) = 58$?

EX #6: Non-potable water is being pumped into a processing center where the depth, in feet, of the water at time t in hours is given as $y = h(t)$. Interpret the following statements using the proper units.

A. $h(7) = 5$

B. $h'(7) = 0.6$

C. $h^{-1}(7) = 9$

D. $(h^{-1})'(7) = 1.4$

Lesson 2: Straight Line Motion
Connecting Position, Velocity & Acceleration

Topic 4.2: Straight Line Motion - Connecting Position, Velocity & Acceleration

In Unit 2, Lesson 7, the big ideas of position, velocity and acceleration were introduced. Particle motion problems deal with motion along a straight line. This includes horizontal and vertical motion of a falling object. It's important to note that this motion means the particle is moving along an axis, not along the graph of the function.

Vocabulary Review

The vocabulary terms for this lesson are restated here for your convenience.

Position	$s(t), x(t), y(t)$ tells where the particle is located at a given time, t.
Average Velocity	The average rate of change over $a \leq t \leq b$ $$v_{avg.} = \frac{\Delta s}{\Delta t} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{s(b) - s(a)}{b - a} = m_{\text{secant}}$$
Instantaneous Velocity	$s'(t) = v(t)$ is rate of change of position at a given time and direction of movement. $$\frac{ds}{dt} = s'(t) = v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$
Speed	The rate at which the position is changing, $\lvert\text{velocity}\rvert = \lvert v(t)\rvert = \left\lvert\frac{ds}{dt}\right\rvert$
Acceleration	$s''(t) = v'(t) = a(t)$ tells how fast the velocity is changing at a given time. The sign tells if velocity is increasing or decreasing. $$s''(t) = \frac{d^2s}{dt^2} = v'(t) = \frac{dv}{dt} = a(t)$$
Displacement over an interval $[a, b]$	$s(b) - s(a)$ is the difference in distance from start to stop.
Distance Traveled	The sum of the absolute values of the distances between turning points.
Free Fall Motion	$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$ is the motion of an object under the influence of gravity ignoring air resistance.
Direction changes	When velocity changes from either positive to negative or negative to positive at a point (time) . Velocity does not need to pass through zero or even be defined at a point where it's sign changes in order for direction to change.
Gravitational Constants	$9.8 \frac{m}{sec^2}$ or $32 \frac{ft}{sec^2}$

The focus of this lesson is to make connections and transition between position, velocity and acceleration functions as they relate to the skills and rigor on the AP Calculus Exam®. We will look at algebraic/ numerical problems, table problems, and graphical interpretations related to straight line motion in this lesson.

Algebraic and Numeric Question Stems

EX #1: A particle moves along the x-axis so that at any time $t \geq 0$, the position is given by the function $x(t) = t^3 - \frac{9}{2}t^2 - 12t + 5$

A. At $t = 0$, determine whether the particle is moving to the right or to the left. Explain how you know.

B. At what time(s) does the particle change directions. Justify your answer.

C. At $t = 1$, is the velocity of the particle increasing or decreasing? Explain your answer.

D. At $t = 1$, is the speed of the particle increasing or decreasing. Justify.

E. What is the particle's acceleration at $t = 1/4$**?** Explain the meaning of your answer in terms of the particle's velocity. How would units be explained in context to the particle?

EX #2: The table below gives the velocity, in feet per second, of a particle moving along a horizontal axis. The velocity v is a twice-differentiable function of time t.

Time, t (sec)	0	2	3	5	7	10	12	15
Velocity, $v(t)$ (ft/sec)	-4	-1	1	2	3	5	3	6

A. At $t = 0$, determine whether the particle is moving to the right or to the left. Explain how you know.

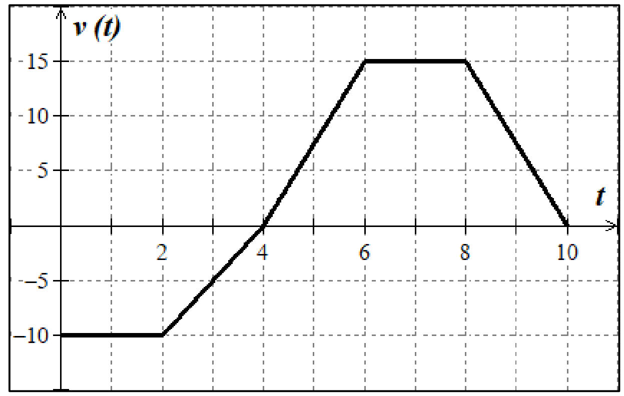
B. Is there any time(s) during the interval $0 \leq t \leq 15$ seconds when the particle is at rest? Explain your answer.

C. Use the data from the table to find an approximation for $v'(8)$. Explain the meaning of $v'(8)$ in terms of the particle motion. Show the computations that lead to your answer, and indicate the units of measure.

D. Find the average acceleration of the particle for $10 \leq t \leq 12$ seconds. Explain what this number means in terms of the particle's velocity on the interval.

E. Let $a(t) = v'(t)$. Is there a time $t = c$ guaranteed such that $a(c) = 0$, in the interval $0 \leq t \leq 15$? Justify your answer.

EX #3: The graph at right is $y = v(t)$, the velocity of an object moving on a line over the time interval $[0, 10]$. Assume that the positive direction is to the right.



A. On what interval of time is the object moving to the right?

B. When is the object speeding up? Explain.

C. How fast is the object moving at time $t = 3$, and in what direction?

D. What is the average velocity over the time interval $0 < t < 4$ and $4 < t < 10$? Show the computations that lead to your answer.

E. What is the total distance traveled by the object? Show the computations that lead to your answer.

F. When is the object slowing down? Justify.

G. When is the velocity increasing? Explain.

H. On what interval(s) does the object move at a constant speed?

EX #4: An object moves along the x-axis for time $0 \leq t \leq 6$. The velocity of the object at time t is given by $v(t) = \ln(t) - 0.15t^3 + \left(t^2 + 2t\right)^{4/5}$.

A. For $t = 4$, is the object speeding up or slowing down? Justify.

B. At what time on $0 \leq t \leq 6$ does the particle change directions? Justify.

EX #5: A particle moves along the x-axis so that at any time $t \geq 0$, its velocity is given in meters per second by the function $v(t) = 4 + 2.3\sin(0.85t)$. What is the acceleration of the particle at $t = 4$? What does this tell you about the direction of the particle?

EX #6: An object moves along a line so that at time t, $0 \leq t \leq \pi$, its position is given by $s(t) = -3\sin\left(0.25t^2\right) + 1$. What is the velocity of the object when its acceleration is zero? Describe the behavior of the object.

EX #7: A particle moves along the x-axis so that at time t, $0 \leq t \leq 15$, the velocity of the particle is given by $v(t) = \tan(\sqrt[3]{t} - 0.2t)$.

A. Find the time t when the acceleration is zero. What does this mean about the velocity at that instant?

B. At time $t = 6$, what is the velocity and acceleration of the particle? Is the particle speeding up or slowing down? Justify.

Lesson 3: Rates of Change in Other Applications

Topic 4.3: Rates of Change in Applied Contexts Other Than Motion

In this lesson we will shift our focus from physics and vertical motion problems into the world of economics, business calculus and other rates of change. When business owners calculate profits for their company, many different factors must be considered. For this course, we want to explore marginal analysis as it relates to quantities such as fixed costs, revenue, and profits. You have seen similar exercises with a linear cost function, but we want to explore scenarios of a non-linear nature. Let's begin by defining some new vocabulary terms.

Vocabulary Basics for Economic Scenarios

Cost function: $C(x) =$ total cost of producing x units

Average Cost function: $\bar{C}(x) = \frac{C(x)}{x}$ includes the fixed operating costs

Revenue function: $R(x) =$ total revenue received for the sale of x units

Profit function: Profit = Revenue − Cost

$$\boldsymbol{\pi(x) = R(x) - C(x)}$$

Note: *In the business world, the Greek letter for "p", that is, π, is commonly used to name profit. This helps to have different variables for price, p, and profit , π.*

Marginal Analysis of Non-Linear Functions

In marginal analysis, we have a method of estimating how revenue, costs and profits change when the input increases by "***one additional item***." If the graph of a non-linear function is not "***curving too rapidly***" near the point of tangency, then the slope of the tangent line can be used to give us a close approximation for how the quantities change in a relatively small neighborhood.

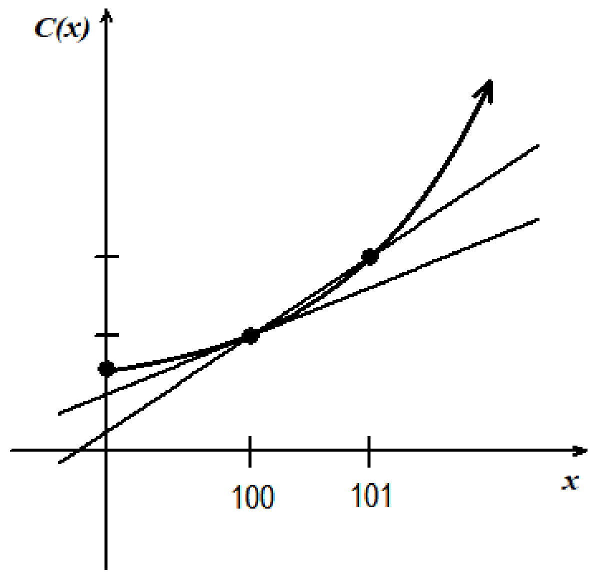
Marginal Cost $\boldsymbol{MC = C'(x)}$ is the rate of change (derivative) of the cost function. It approximates the cost to produce one more item. That is, the "***additional cost at the margin***." To find an **exact amount** to produce one more item, use the **original cost function** , $C(x+1) - C(x)$.

EX #1: The cost to translate documents into another language is a function of the number of pages. The cost function is given by: $C(p) = 0.002p^3 - 0.1p^2 + 8.75p$. Calculate the marginal cost at the production (translation) level of 10 pages, explain what this means in context to the scenario. How does this compare to the exact cost to translate 10 pages?

Let's get a visual interpretation for the cost function.

Non-Linear Cost Function

EX #2: Use the graph, at right, to show the relationships for average cost and marginal cost.



EX #3: The cost of producing a set of Bluetooth headphones is given by the production manager as $C(x) = 1500 + 50x + 0.01x^2$.

A. Find the marginal cost function. Then, use this function to estimate how fast the cost is going up if the company produces 100 sets of headphones.

B. Compare your result from part (A) to the **exact cost** of producing the 101^{st} set of Bluetooth headphones.

C. Find the average cost function $\bar{C}(x)$ and evaluate $\bar{C}(100)$. Explain the meaning in context to the scenario.

EX #4: Suppose those new Bluetooth headphones from EX #3 sold for $125.00.

A. Write the revenue function.

B. Write the profit function.

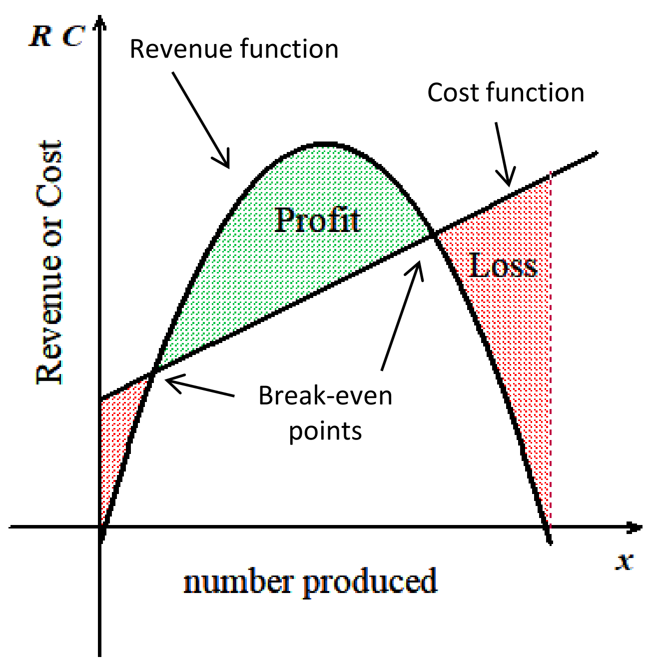
C. Find the marginal revenue function.

D. Compute the marginal profit function if 100 sets of headphones are produced and sold. Interpret your findings.

E. How many headphones should the company produce in order to maximize their profits?

Marginal Cost, Revenue, and Profit

In the graph shown, at right, notice the break-even points occur where the revenue equals the cost.



Maximum Profits

A company would obviously be most interested to know how to maximize profits.

The maximum (or minimum) profit **can occur where *MC = MR.***
Marginal Cost = Marginal Revenue
$C'(x) = R'(x)$.

However, maximum or minimum profits do not necessarily have to occur where *MR=MC. They can also occur at endpoints.*

Consider the slopes of the tangent lines, then when $MC < MR$, a company will make more in extra revenue than it will spend in extra costs by increasing production.

EX #5: An office supply manufacturer produces spiral notebooks for interactive student notebooks. Find the quantity x that maximizes profit if the total revenue, $R(x)$, and the total cost $C(x)$, are given in dollars by

$$R(x) = 8x - 0.004x^2$$
$$C(x) = 700 + 1.2x$$

Where $0 \leq x \leq 1000$ notebooks. What production level gives the maximum profit?

EX #6: The table below shows cost and revenue for the production of a widget. Use the table to answer the following questions.

$\boldsymbol{x}$	0	200	400	600	800	1000
$\boldsymbol{R(x)}$	0	600	1200	1800	2400	3000
$\boldsymbol{C(x)}$	800	1050	1200	1375	1450	1850

A. What is the price of the widget?

B. What are the fixed costs?

C. At what production level, x, is the profit for the widget maximized? Explain your reasoning.

EX #7: When a backpack production is 5000, marginal revenue is \$5 per backpack and marginal cost is \$4.65 per backpack. Would a maximum profit be expected to occur when production levels are above or below 5000 backpacks? Explain.

The last examples in our lesson will be focused on other rates of change that are non-motion related.

EX #8: Gaby is working on a project that watches the decline in great white sharks in the Atlantic Ocean. The function $S(t) = 3570\, e^{-0.2t}$ gives the number of sharks t years since she began working on the project. Find the rate of change for the number of sharks in the oceans after 3 years.

EX #9: Omar uploaded a funny video of sugar gliders on his Instagram account. The video went viral rapidly. The number of views t days after Omar uploaded the video can be modeled by the function $V(t) = 243e^{0.5t}$. What is the instantaneous rate of change of the number of views 5 days after he uploaded his video?

EX #10: Use the following table of values for the number $V(t)$ of gallons of water (in thousands) per hour, h , that are flowing into a retention pond t hours after a severe thunderstorm.

t	0	1	1.5	2	2.5	3	3.5	4
$V(t)$	0.5	0.75	0.93	1.15	1.24	1.05	0.97	0.72

A. What is the interpretation of $V'(t)$?

B. Estimate $V'(2)$. Explain the meaning in context to the problem.

C. When does the rainwater flow begin to slow down? Estimate the instantaneous rate of change at this time.

Lesson 4: Related Rates

Topic 4.5: Solving Related Rates Problems

The derivative, $\frac{dy}{dx}$, of a function, $y = f(x)$, is its instantaneous rate of change with respect to the variable x.

- ☐ When a function describes either position or distance, its rate of change is interpreted as velocity.
- ☐ In general, a time rate of change answers the question: *How fast is a quantity changing?*
  - For example, if V is volume that is changing in time then $\frac{dV}{dt}$ is the rate, or how fast, the volume is changing with respect to time.
  - If a person is walking toward a street-lamp at a constant rate of 3 feet per second, then we know that the distance is decreasing, so $\frac{dx}{dt} = -3\,\frac{ft}{sec}$.
  - If they walk away from the lamp then the distance is increasing and the rate of change becomes positive or $\frac{dx}{dt} = 3\;\frac{ft}{sec}$.

GUIDELINES FOR SOLVING RELATED RATE PROBLEMS:

1. Make a sketch and label the quantities.
2. Read the problem and identify all quantities as: "KNOW", "GIVEN", and "FIND" with the appropriate information.
3. Write an equation involving the variables whose rates of change either are given or are to be determined.
4. Using the Chain Rule, implicitly differentiate both sides of the equation ***with respect to time, t.***
5. *AFTER COMPLETING STEP 4*, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

EX #1: Suppose x and y are both differentiable functions of t and are related by the equation $y = x^2 - 3x$. Find $\frac{dy}{dt}$ when $x = 3$ given that $\frac{dx}{dt} = 2$, when , $x = 3$.

EX. #2: A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

1. KNOW:

2. GIVEN:

3. FIND:

EX. #3: Air is being pumped into a spherical balloon at a rate of 800 cubic centimeters per minute. How fast is the radius of the balloon changing at the instant the radius is 30 centimeters?

1. KNOW:

2. GIVEN:

3. FIND:

EX. #4: The top of a 25- foot ladder is sliding down a vertical wall at a constant rate of 3 feet per minute. When the top of the ladder is 7 feet from the ground, what is the rate of change of the distance between the bottom of the ladder and the wall?

1. KNOW:

2. GIVEN:

3. FIND:

EX. #5: At noon, ship A is 150 km east of ship B. Ship A is sailing west at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4 p.m.?

1. KNOW:

2. GIVEN:

3. FIND:

EX #6: A right circular cylinder is changing shape. The radius is decreasing at a rate of 2 inches/sec while its height is increasing at the rate of 5 inches/ sec. When the radius is 4 inches and the height is 6 inches, how fast is the volume changing?

Given: $V = \pi r^2 h$

1. KNOW:

2. GIVEN:

3. FIND:

EX #7: An inverted cone is leaking water at a rate of 1 cm^3/min. The cone has a height of 9 cm and a diameter of 6 cm. Find the rate at which the water level is dropping when $h = 3\ cm$.

Given: $V = \frac{1}{3}\pi r^2 h$

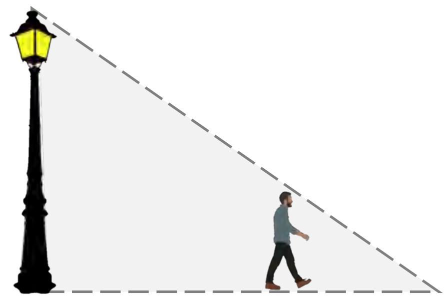
1. KNOW:

2. GIVEN:

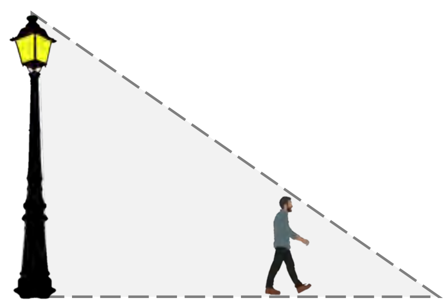
3. FIND:

EX #8: A lamppost is 26-feet above the sidewalk. Andy Grate is a 6-foot tall Calculus teacher and is walking away from the lamp at a rate of 4 feet per second. When he is 20 feet away from the lamppost, find each of the following.

A. How fast is the length of his shadow changing?



B. At what rate is the tip of his shadow moving?



Lesson 5: Linear Approximation

Topic 4.6: Approximating Values of a Function Using Local Linearity and Linearization

In Unit 2 we briefly looked at the idea of local linearity, when zooming in on a function, we discovered as we zoom in on a function $y = f(x)$, the straight line that appears at a given point $x = a$ has a slope equal to the derivative $f'(a)$, and the equation for the tangent line at the given point was defined as:

$$f(x) = f(a) + f'(a)(x - a)$$

In this lesson, we will expand our investigation for a deeper understanding.

The Tangent Line Approximation

Suppose f is a differentiable function at a point $x = a$. Then, for values of x near a, the tangent line approximation to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a)$$

We will define the expression $f(a) + f'(a)(x - a)$ as the ***local linearization of f near*** $\boldsymbol{x = a}$.

If you consider that point a is fixed, then $f(a)$ and $f'(a)$ are constant.

So , $f(x) \approx L(x) = f(a) + f'(a)(x - a)$ if x is close to a.

Graphically, linear approximation is $\Delta f \approx f'(a)\,\Delta x$, where $\Delta f = f(a + \Delta x) - f(a)$

The Linear Approximation Error

The **error**, $\boldsymbol{E(x)}$, in the linear approximation is simply the vertical "gap" between the actual curve and the tangent line. The error can be calculated by:

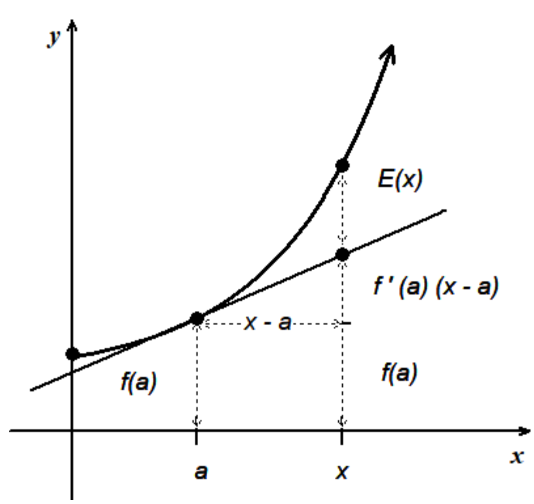
Error = |Exact − Approximation |

$$E(x) = |\Delta f - f'(a)\,\Delta x|$$

$$\text{Percentage Error} = \left|\frac{\text{exact} - \text{approximation}}{\text{exact}}\right| \times 100\%$$

Visual Vocabulary

EX #1: It can be shown that the best linear approximation to f near a is the tangent line approximation. Label the true value of $f(x)$, the tangent line , and the approximation.



Procedure for Finding a Linear Approximation for f(x)

So, if your head is spinning with this new information, let's summarize what we will be focused upon for the remainder of the lesson. When you need to write a linear approximation for $f(x)$ centered at a given value, say $x = c$, and you need to approximate $f(x)$ at a value that is in the neighborhood of $x = c$, let's call this new value $x = a$, then proceed as follows:

1. Write the **point-slope equation** for the tangent line at $(c, f(c))$.
2. Isolate y and **rename** this equation as $L(x)$.
3. **Substitute** your value for $x = a$ in the equation, $L(x)$. Be sure to write the notation correctly:
$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$
4. If you need to explain whether or not $L(a)$ is an **overestimate** or an **underestimate**, consider the behavior of the function in your local neighborhood.
   - ❑ If the function is **concave up** at $x = c$, $L(a)$ is an **underestimation**.
   - ❑ If the function is **concave down** at $x = c$, $L(a)$ is an **overestimation**.
5. When **finding the error** or accuracy of an approximation, use your calculator. Find the ***absolute value of the difference between the exact amount and your approximation***.
   Then, the error should be stated as being less than 10^{-p}, where p is the number of zeros in your result.
   For example: $|\text{Error}| < 10^{-3}$ would be the reported error for a result such as, 0.000147
6. If you need to find the Percentage Error = $\left|\frac{\text{exact } - \text{approximation}}{\text{exact}}\right| \times 100\%$

EX #2: The function f is twice differentiable where $f(3) = -2$, $f'(3) = 5$ and $f''(3) = 1$. Write the equation of the tangent line. Then, use the equation to find the value of $f(3.1)$.

Linear approximations will do a great job of estimating values for $f(x)$, provided you stay in the small "neighborhood" near $x = a$. The farther away from $x = a$ we move, the worse the approximation becomes. While there is no easy method to determine how near or far away we can move from our point of tangency, we can predict the amount of error to give us an idea of how "good" the approximation models the actual value.

Let's look at a famous linear approximation that is used to describe vibrations, pendulum motion, and is used as a method to simplify formulas in optics.

EX #3: Consider $y = \sin x$ at $x = \frac{\pi}{3}$.

We know the ordered pair corresponding to this angle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

A. Since the angle $\frac{\pi}{3} \approx 1.0471975\ldots$ Let's write the linear approximation for $x = 1.05$.

B. Find the error of your approximation.

EX #4: Find a linear approximation to $y = 2^x$ at $x = 3$. Use the tangent line to estimate $x = 2.97$

EX #5: Approximate $(2.01)^5$. Use your calculator to find the accuracy of the approximation. Is the linearization an overestimate or underestimate? Explain.

EX #6: Find the linear approximation to $g(x) = \sqrt[3]{x}$ at $x = 8$. Use the linear approximation to approximate the value of $\sqrt[3]{10}$ and $\sqrt[3]{16}$. Compare the approximate values to the exact values. Find the error of your approximations.

EX #7: Let f be a function that is differentiable for all real numbers. The table below gives the values of $f(x)$ and its derivative $f'(x)$ for selected values in the interval $[-1.0, 1.0]$. The function is **concave up** in this closed interval.

x	−1.0	−0.8	−0.6	−0.3	0	0.3	0.6	0.8	1.0
$f(x)$	−12	−36	−51	−64	−87	−73	−67	−19	5
$f'(x)$	−8	−5	−2	−1	0	2	3	6	14

A. Write an equation of the line tangent to the graph of $f(x)$ where $x = -0.6$.

B. Use this line to approximate the value of $f(-0.5)$.

C. Is this approximation greater or less than the actual value of $f(-0.5)$ and give a reason.

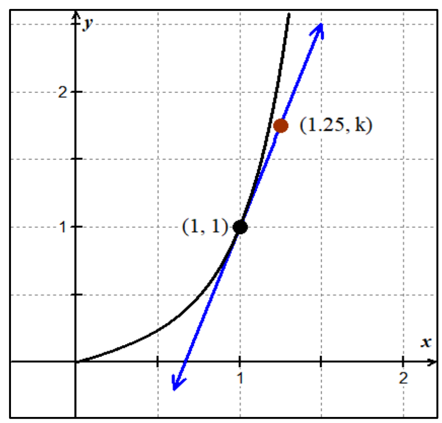
EX #8: Write the linear approximation equation for $f(x) = \frac{1}{x-1}$ at $x = -1$.

A. Find an approximation for $f(-1.2)$

B. Calculate the error of the approximation.

EX #9: Given the implicit curve, $x^2 + \ln\left(\frac{x}{y}\right) = 1$;

A. Write the linear approximation of the tangent line at $x = 1$.



B. Using the tangent line to the curve at $x = 1$, find the number k such that the point $(1.25, k)$ is on the curve.

C. Calculate the error of the approximation.

Lesson 6: L'Hospital's Rule

Topic 4.7: Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

L'Hospital's Rule is a great tool for finding limits that are often difficult to evaluate. It can also help with determining limits at infinity for "asymptotic behavior." We will use it for curve sketching in the next section, as well.

When evaluating limits by direct substitution, there are seven expressions known as **indeterminate form**. These expressions are

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^{\infty}, 0^{0}, \infty - \infty, \infty^{0}$$

Consider the limit of a quotient: $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$.

L'Hospital's Rule states that *when* $f(x)/g(x)$ *has an indeterminate form of the type* $0/0$ *or* ∞/∞ *at* $x = c$*, then we can replace* $f(x)/g(x)$ *by the quotient of the derivatives* $f'(x)/g'(x)$.

L'Hospital's Rule

Assume $f(x)$ and $g(x)$ are differentiable on an open interval (a, b) containing c. Also assume that $g'(x) \neq 0$ for all x in (a, b); except possibly at c itself. Then, if

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0 \quad \text{; then}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

If the limit on the right exists or is infinite (∞ or $-\infty$). This conclusion also holds if $f(x)$ and $g(x)$ are differentiable for x near (but not equal to) c and

$$\lim_{x \to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \pm\infty$$

This rule is also valid for one-sided limits.

A Word of Caution:

In order to be able to use L'Hospital's Rule; it is important to note that these conditions are met:

1. $g'(c) \neq 0$; If the denominator is zero then the limit is undefined.
2. No oscillating functions, as x approaches c, the limit must approach a single y-value.
3. $\lim\limits_{x \to c} \left(\frac{f(x)}{g(x)}\right) \neq \pm\infty$; if the limit approaches infinity as x approaches c, the limit is undefined.

Communication in mathematics is an important skill. Let's turn our discussion to the proper notation and protocol for using L'Hospital's Rule correctly on the newly redesigned AP Calculus exams before we practice.

Proper Notation and Protocol

In order to use L'Hospital's Rule on the AP Calculus Exam® students should show proper procedural steps and notation. To master the process be sure to . . .

1. Explicitly show the limit of the numerator is 0, ∞, or $-\infty$.

2. Explicitly show the limit of the denominator is 0, ∞, or $-\infty$.

3. You may want to state that the limit is indeterminate form because they are both 0 or $\pm\infty$.

4. By L'Hospital's Rule, write the limit as $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$

5. Find the limit using proper limit notation throughout.

Type I: Simple Polynomials and Powers

EX #1: $\lim\limits_{x\to 5} \dfrac{2x^2 - 9x - 5}{x - 5}$

EX #2: $\lim\limits_{x\to -5} \dfrac{\sqrt{4 - x} - 3}{x + 5}$

Type 2: Repeated Use of L'Hospital's Rule

EX #3: $\lim\limits_{x \to -\infty} \dfrac{x^2}{e^{-x}}$

EX #4: $\lim\limits_{x \to -2} \dfrac{x^3 + x^2 - 8x - 12}{x^3 + 8x^2 + 20x + 16}$

Type 3: Trigonometry and Transcendental Functions

EX #5: $\lim\limits_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^2}$

EX #6: $\lim\limits_{x\to 1} \dfrac{\ln x - x + 1}{e^x - e}$

Type 4: L'Hospital's Rule for $\pm\dfrac{\infty}{\infty}$

EX #7: $\lim\limits_{x\to \infty} \dfrac{9x^2 + 10x - 3}{12x^2 - 4}$

EX #8: $\lim_{x \to \infty} \dfrac{4x^2 - 3x + 2}{e^{2x} + \ln x}$

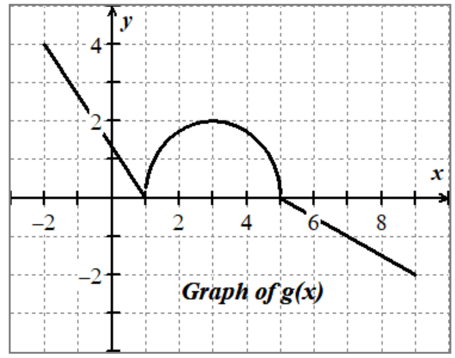
Type 5: AP Style Applications of L'Hospital's Rule

EX #9: The table below gives values of a twice-differentiable function $y = f(x)$. Use the table to find the limit, if it exists. Justify your answer.

$$\lim_{x \to -2} \frac{f(3x+2)}{x^2 - 4}$$

x	-4	-1	2
$f(x)$	0	3	0
$f'(x)$	8	-2	6

EX #10: Use the graph of $y = g(x)$, shown below to find the following limit, if possible. Use proper notation to justify your findings.



$$\lim_{x \to 3} \frac{g(x)}{x - 3}$$

EX #11: The differential equation for the curve $y = h(x)$is given by $\frac{dy}{dx} = \frac{3x}{y+2}$. Given that h is twice-differentiable and $h(3) = -1$, find:

$$\lim_{x \to 3} \frac{h(x) + x - 2}{\ln(7 - 2x)}$$

EX #12: Find the limit, if possible. Give a reason for your answer.

$$\lim_{x \to \infty} \frac{2^x}{x^{99}}$$