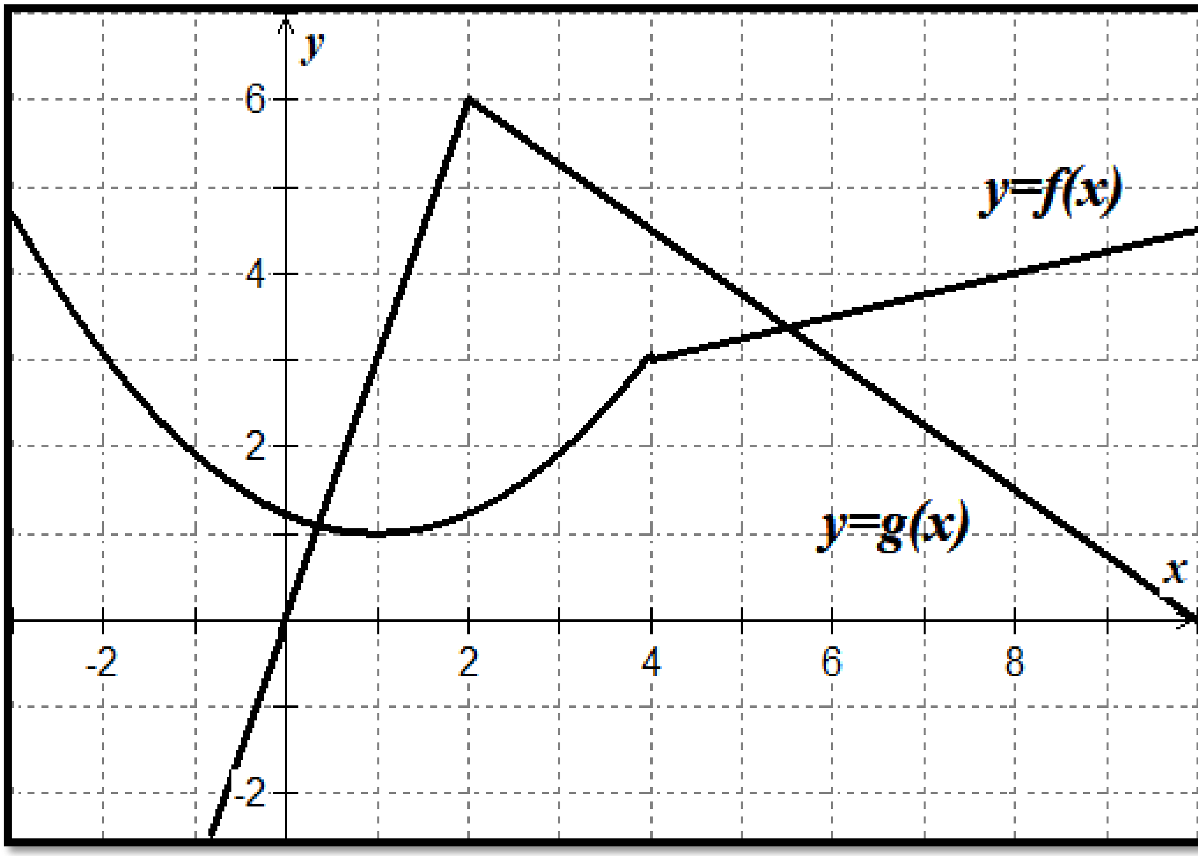


Name: ______________________________

UNIT 3:

DIFFERENTIATION

Composite, Implicit, & Inverse Functions

Part 2

NOTES

Lesson 1: The Chain Rule

Topic 3.1: The Chain Rule

The chain rule is used to differentiate composite functions such as $y = \sqrt{x^3+2}$ or $y = \sin(x^2)$. Remember that you create a composite function by "plugging" one function into another function. The composition of f and g is denoted by $(f \circ g)(x) = f(g(x))$. We can find the derivative of each separate function by using the rules we've learned so far. But, we need the Chain Rule to find the derivative of two functions "plugged into one another."

The Chain Rule:

If f and g are differentiable, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable and $f'(g(x)) \cdot g'(x)$

Leibniz notation will allow us to use "u-substitution." Let $y = f(u)$ and $u = g(x)$, notice that y is a function of u and u is a function of x. So it follows that,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This looks more complicated than it really is. Basically, the Chain Rule says to:

MULTIPLY THE DERIVATIVE OF THE INSIDE FUNCTION BY THE DERIVATIVE OF THE OUTSIDE FUNCTION.

EX #1: Find the derivative of $f(x) = (5x+3)^2$ with and without the chain rule.

A. Without the chain rule	**B.** With the chain rule

EX #2: Find $f'(x)$ given $f(x) = (4 - x^2)^3$

$f'(x) =$ (**D**erivative of **O**utside **F**unction) × (**D**erivative of **I**nside **F**unction)

General Power and Exponential Rules:

If $g(x)$ is differentiable, then

- ❑ $\frac{d}{dx}\, g(x)^n = n\left(g(x)\right)^{n-1}\, g'(x)$ (for any number n)
- ❑ $\frac{d}{dx}\, e^{g(x)} = g'(x)e^{g(x)}$

EX #3: For each function below, find the derivative.

A. $f(x) = \sqrt{(x^2-1)^3}$	**B.** $f(x) = \frac{-7}{(2x-3)^2}$
C. $y = \left(5-4x^2\right)^{2/3}$	**D.** $y = e^{3x^2}$
E. $y = \frac{-2}{\sqrt[3]{6x+3}}$	**F.** $y = \cos^3\left(e^{5\theta}\right)$

G. $f(x) = \ln\left(x^2 + x + 1\right)^2$

H. $f(x) = x^2 e^{-4 \ln x}$

I. $g(x) = cos^2 4x - sin^2 4x$

J. $y = x^4 \sin^2 x$

K. $H(x) = ln\ (ln\ x)$

L. $f(x) = sin\ (\ e^{\pi x})$

M. $g(\theta) = 4cos^2(\pi\theta)$

N. $y = \tan\sqrt{2x - 3}$

More Derivatives Using Tables

EX #4: Let $f(x)$ be a continuous and differentiable function on the interval $0 \le x \le 1$, and let $g(x) = f(4x)$. The table below gives values of $f'(x)$, the derivative of $f(x)$. Find the value of $g'(0.2)$.

$\boldsymbol{x}$	0.1	0.2	0.4	0.5	0.6	0.7	0.8
$\boldsymbol{f'(x)}$	0.75	0.823	0.964	1.033	1.081	1.104	1.125

EX #5: Use the table below to complete the following.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	-2	4	-1	2
0	4	3	2	-3
2	3	-2	-4	1

A. Find the equation of the tangent line to $x = -1$, if $H(x) = 3\, f\big(g(x)\big)$

B. If $J(x) = f\big(g(x)\big) \cdot \cos x$, what is the value of $J'(0)$?

Lesson 2: Differentiation by Tables, Graphs and Symbolically

Topic 3.5: Selecting Procedures for Calculating Derivatives

There will be times when you need to find an approximation of the derivative. Typically, this type of question will be given in the form of a table of values without a specific function. You will need to identify an appropriate mathematical rule or procedure based on the given data to proceed successfully.

EX #1: The table gives the values of a function that is differentiable on the interval [0, 1]. Find $f'(0.10)$.

x	0.10	0.20	0.30	0.40	0.50
$f(x)$	0.261	0.433	0.552	0.587	0.534

EX #2: Symbolic Differentiation

Given the table of values below, find each of the derivatives of the following at $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	-4	8	3	-1
3	5	-2	1	-3
4	0	-1	2	3

A. $H(x) = f(g(x))$

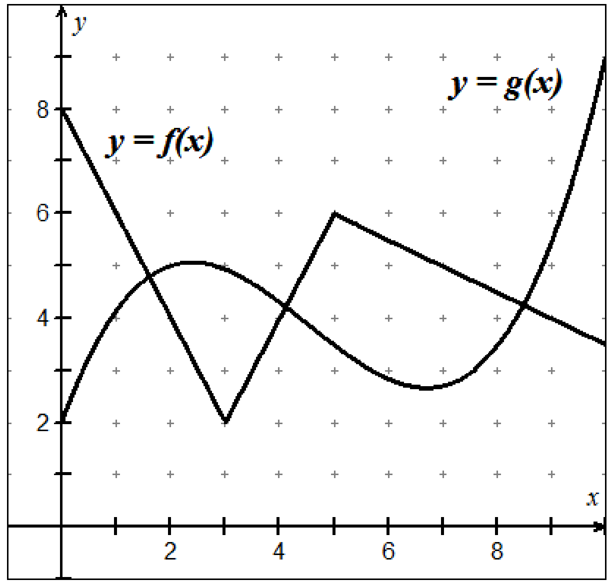
B. $K(x) = [f(x)]^3$

C. $P(x) = f(x) \cdot g(2x)$

D. $R(x) = \frac{f(2x)}{g(x)}$

EX #3: Finding derivatives from a graph.

Let $H(x) = f\left(g(x)\right)$, where the graphs of f and g are shown at right. Estimate each derivative. Justify your analysis algebraically.



A. $g'(3)$

B. $f'\left(g(3)\right)$

C. $H'(3)$

D. $f'(7)$

E. $g'\left(f(7)\right)$

F. $H'(7)$

Lesson 3: Implicit Differentiation

Topic 3.2: Implicit Differentiation

We have been able to differentiate functions that are solved for y explicitly, up to this point. Now we want to consider functions of the type $x^2 - 2y^3 + 4y = 2$. You can see that it would be quite challenging to solve for y as a function of x, explicitly.

Some basic facts to consider related to implicit differentiation:

- ❑ Realize differentiation is taking place with respect to x.
- ❑ When you differentiate terms involving x alone, you can differentiate as usual.
- ❑ When you differentiate terms involving y, you must apply the Chain Rule (because you are assuming that y is defined implicitly as a differentiable function of x.
- ❑ $\frac{d}{dx}[y] = \frac{dy}{dx}$
- ❑ $\frac{d}{dx}[x] = 1$

Guidelines for Implicit Differentiation:

1. Differentiate **both sides** with respect to x. *OBEY THE CHAIN RULE* by multiplying by dy/dx every time you differentiate an expression containing y.
2. Isolate dy/dx by getting all of the dy/dx terms onto one side of the equation, and all other terms onto the other side.
3. Factor out dy/dx if necessary.
4. Solve for dy/dx.

EX #1: Find $\frac{dy}{dx}$ for $xy + y = 8$ at (3,2)

EX #2: Differentiate $x + 3xy - 2y^2 = 2$ at $(1, 1)$

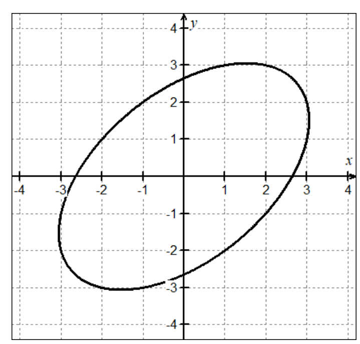
EX #3: Find $\frac{dy}{dx}$: $y^3 + 5y^2 - 5y - x^2 = -4$

EX #4: Find y' for $ye^x = 3\pi$

EX #5: Differentiate: $\cos xy = x + 3y$

EX #6: Differentiate: $\ln xy = \sin x$

EX #7: Given $x^2 - xy + y^2 = 7$



A. Find the equation of the tangent line at $x = (-1,2)$**.**

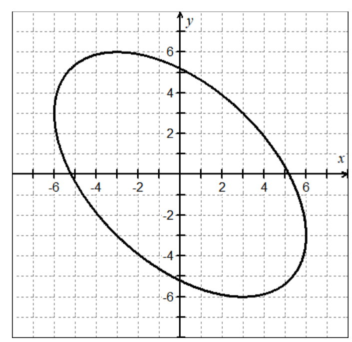
B. Find the equation of the normal line at $x = (-1, 2)$

Topic 3.6: Calculating Higher-Order Derivatives

When finding higher-order derivatives using implicit differentiation, be sure to solve for $\frac{dy}{dx}$ before finding the second derivative, or higher. The subsequent derivatives will have a $\frac{dy}{dx}$ term. For each of these terms, you can make a substitution to be able to put your final derivative in terms of x and y.

EX. #8: Find $\frac{d^2y}{dx^2}$ given $\cos y = x$

EX #9: Find the points at which the graph of $x^2 + xy + y^2 = 27$ has a either a horizontal or vertical tangent line.



Lesson 4: Derivatives of Inverse Functions

Topic 3.3: Differentiating Inverse Functions

Recall from previous courses that a function, $y = f(x)$, that is **one-to-one** will pass the **horizontal line test** and will therefore, have a unique **inverse function** $y = f^{-1}(x)$. We often say that the inverse function "undoes" the original function $f(x)$. This can be seen in the test to prove two functions are inverses of each other, since the composition of the two functions produces the identity function.

$$f\left(f^{-1}(x)\right) = x = f^{-1}\left(f(x)\right)$$

Definition of an Inverse:

A function g is the inverse function of the function f if $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f. The function $g(x)$ is denoted by $f^{-1}(x)$.

Reflective Property of Inverse Functions:

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

EX #1: Use the reflective property of inverse functions and the function values for $f(x)$ and $g(x)$ in the table at right, to complete the table of values for their respective inverse functions, $f^{-1}(x)$ and $g^{-1}(x)$. Then, use them to answer each of the questions below.

x	$f(x)$	$g(x)$
3	2	−1
1	−1	0
−1	4	3

A. The table of values for $f^{-1}(x)$:

x	$f^{-1}(x)$

B. The table of values for $g^{-1}(x)$:

x	$g^{-1}(x)$

Use the tables above to evaluate each of the following expressions below:

C. $f\left(g^{-1}(0)\right)$

D. $f^{-1}\left(f(-1)\right)$

E. $f^{-1}\left(g^{-1}(-1)\right)$

F. $g^{-1}\left(f^{-1}(4)\right)$

The Existence of an Inverse Function:

1. A function has an inverse function if and only if it is one-to-one.

2. If f is strictly monotonic on its entire domain, (either increasing or decreasing on its entire domain) then it is one-to-one and therefore has an inverse function.

The relationship between a composite function and its inverse is stated below. If you use the chain rule and differentiate both sides, you can find a formula for the derivative of an inverse.

$$f[f^{-1}(x)] = x$$

EX #2: Given $f(x) = 4x - 3$ and $f'(2) = 4$. Find the value of $[f^{-1}(5)]'$

GOAL: How do we find the derivative of the inverse function of $f(x)$ at $x = k$?

To Find the Derivative of an Inverse Function:

Let f and g be inverse functions, such that $f\left(g(x)\right) = x = g\left(f(x)\right)$ where $f(a) = b$ and $g(b) = a$.

To find $g'(b)$ for a point (a, b) on $f(x)$.

1. Find $f'(x)$
2. If you are only given b, set $b = f(x)$ to find a.
3. Find $f'(a)$
4. $g'(b) = \frac{1}{f'(a)}$

NOTE: $\left(f^{-1}\right)'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$

****** Inverse functions have reciprocal slopes at corresponding points.******

EX #3: Find $g'(-1)$; where $g(x)$ is the inverse of $f(x)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	5	0		
3	-1	-4		
-1	4	6		

EX #4: Let g be the inverse of f. Calculate $g'(-9)$ for $f(x) = 4x^3 + 5x$

There are questions where you will need to use a calculator to find approximations accurate to three decimal places. Here's an example of such a task.

EX #5: If $f(x) = 4x^7 - x^2 + x - 5$, find $(f^{-1})'(3)$ accurate to three decimal places.

1. Solve $f(x) = 3$, this will give you $x = a$
2. Differentiate $f(x)$
3. Evaluate $f'(a)$ at the value from step 1, using all the decimal values, no early rounding.
4. Use your result in the inverse derivative formula $g'(b) = \frac{1}{f^{-1}(a)}$

Derivatives of Other Bases

***Let* a *be a positive real number (a ≠ 1)* and let *u* be a differentiable function of *x*.**

1. $\frac{d}{dx}[a^x] = (ln\ a)a^x$

2. $\frac{d}{dx}[a^u] = (ln\ a)a^u \frac{du}{dx}$

3. $\frac{d}{dx}[\log_a x] = \frac{1}{(ln\ a)x}$

4. $\frac{d}{dx}[\log_a u] = \frac{1}{(ln\ a)u}\frac{du}{dx}$

EX #6: Find the derivative of each of the following.

A. $y = 3^x$

B. $y = 4^{3x}$

C. $y = 5^{x^2}$

D. $y = 5^{3-x^2}$

EX #7: Find the derivative of each of the following.

A. $y = \log_5 x$	**B.** $y = \log_3(sin\ x)$
C. $y = \log(x + 3^x)$	**D.** $y = \log_2 \sqrt[3]{2x + 1}$

EX #8: Find an equation of the tangent line of $f(x) = 6^x$ where $x = 2$.

EX #9: A bacteria is growing according to the function $W(t) = \frac{1.3}{1+0.3e^{-.4t}}$ for $t \geq 0$ where $W(t)$ is the weight of the culture in ounces and t is in hours. Find the weight of the culture after 0 hour, 2 hours, 8 hours. What is the limit as t approaches infinity?

EX #10: Inverse Derivatives by Tables

Use the table at right to find each of the indicated values below. State the ordered pairs for the inverse function $g^{-1}(x)$ in the column provided.

x	f	g	f'	g'	g^{-1}
-3	2	5	0	4	
0	5	-1	-5	1	
2	4	-3	1	2	
4	2	2	-1	-3	

A. Find $[f^{-1}(4)]'$

B. Find $[g^{-1}(-3)]'$

C. Write the equation for the line tangent to the graph of $g^{-1}(x)$ when $x = 2$.

D. Estimate the value of $g'(3)$**.** Use the result to explain the behavior of the graph of the function g when $x = 3$.

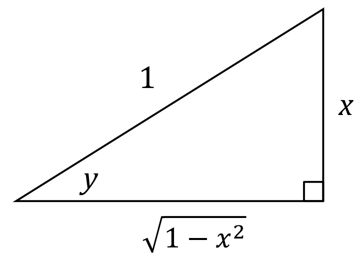
Lesson 5: Derivatives of Inverse Trigonometric Functions

Topic 3.4: Differentiating Inverse Trigonometric Functions

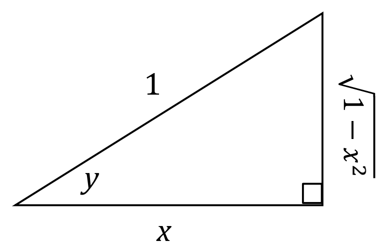
In this final lesson for the unit we can take the derivative of an inverse trigonometric function using the ratios of a right triangle and implicit differentiation. We will start with sine inverse. Let's derive!

EX #1: Use the Pythagorean Theorem, properties of right triangle trigonometry and implicit differentiation to derive the inverse trigonometric function formulas.

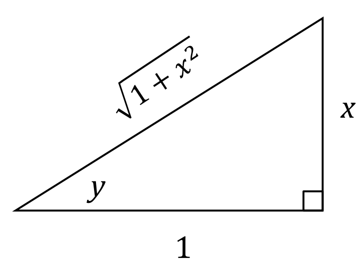
A. Given $y = sin^{-1}x$, find $\frac{d}{dx}(\sin^{-1} x)$



B. Given $y = cos^{-1}x$, find $\frac{d}{dx}(\cos^{-1} x)$



C. Given $y = tan^{-1}x$, find $\frac{d}{dx}(\tan^{-1} x)$



Derivatives of Inverse Trigonometric Functions

NOTEWORTHY: On previous exams, it is most important to know the inverse formulas for sine, cosine, and tangent. You can memorize them or learn to derive them!

$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$
$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$	$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$
$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$	$\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$

EX #2: Differentiate.

A. $y = \sin^{-1}(5x)$

B. $y = \tan^{-1}\left(\sqrt{2x+1}\right)$

C. $y = arccos\left(x^2\right)$

D. $y = \sin^{-1}(x^2-1)$

E. $y = \tan^{-1}(\cos x)$

F. $y = sin^{-1}(e^x)$

EX #3: Find an equation of the tangent line to the graph of the function $y = \arctan\left(\frac{x}{2}\right)$ at the point $\left(2, \frac{\pi}{4}\right)$.

EX #4: Find $\frac{dy}{dx}$ when $\tan(4x - y) = 4x$. [*Hint: all trig functions will disappear in your answer.*]

EX #5: Let g be the inverse of the function $f(x) = 2\cos x, \quad 0 \le x \le \pi$. Find the value of $g'(1)$.

EX #6: Find the slope-intercept form of the tangent line at $\left(\frac{\pi}{4}, -1\right)$ to the graph of $x^2 + x\tan^{-1} y = y - 1$. Round to 3 decimal places.