Section 8.7 Indeterminate Forms and L'Hôpital's Rule

- **Recognize limits that produce indeterminate forms.**
- **Apply L'Hôpital's Rule to evaluate a limit.**

Indeterminate Forms

Recall from Chapters 1 and 3 that the forms $0/0$ and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist. When you encountered one of these indeterminate forms earlier in the text, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate Form	*Limit*	*Algebraic Technique*
$\frac{0}{0}$	$\lim_{x\to -1} \frac{2x^2-2}{x+1} = \lim_{x\to -1} 2(x-1) = -4$	Divide numerator and denominator by $(x+1)$.
$\frac{\infty}{\infty}$	$\lim_{x\to\infty} \frac{3x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{3-(1/x^2)}{2+(1/x^2)} = \frac{3}{2}$	Divide numerator and denominator by x^2.

Occasionally, you can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x\to 0} \frac{e^{2x}-1}{e^x-1}$$

produces the indeterminate form $0/0$. Factoring and then dividing produces

$$\lim_{x\to 0} \frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0} \frac{(e^x+1)(e^x-1)}{e^x-1} = \lim_{x\to 0} (e^x+1) = 2.$$

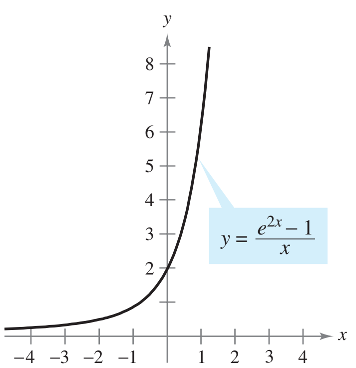
However, not all indeterminate forms can be evaluated by algebraic manipulation. This is often true when *both* algebraic and transcendental functions are involved. For instance, the limit

$$\lim_{x\to 0} \frac{e^{2x}-1}{x}$$

produces the indeterminate form $0/0$. Rewriting the expression to obtain

$$\lim_{x\to 0} \left(\frac{e^{2x}}{x} - \frac{1}{x}\right)$$

merely produces another indeterminate form, $\infty - \infty$. Of course, you could use technology to estimate the limit, as shown in the table and in Figure 8.14. From the table and the graph, the limit appears to be 2. (This limit will be verified in Example 1.)



The limit as x approaches 0 appears to be 2.
Figure 8.14

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\frac{e^{2x}-1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389

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GUILLAUME L'HÔPITAL (1661–1704)

L'Hôpital's Rule is named after the French mathematician Guillaume François Antoine de L'Hôpital. L'Hôpital is credited with writing the first text on differential calculus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "… I acknowledge that I owe very much to the bright minds of the Bernoulli brothers. … I have made free use of their discoveries …," said L'Hôpital.

L'Hôpital's Rule

To find the limit illustrated in Figure 8.14, you can use a theorem called **L'Hôpital's Rule.** This theorem states that under certain conditions the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$

To prove this theorem, you can use a more general result called the **Extended Mean Value Theorem.**

THEOREM 8.3 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

NOTE To see why this is called the Extended Mean Value Theorem, consider the special case in which $g(x) = x$. For this case, you obtain the "standard" Mean Value Theorem as presented in Section 3.2.

The Extended Mean Value Theorem and L'Hôpital's Rule are both proved in Appendix A.

THEOREM 8.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

NOTE People occasionally use L'Hôpital's Rule incorrectly by applying the Quotient Rule to $f(x)/g(x)$. Be sure you see that the rule involves $f'(x)/g'(x)$, not the derivative of $f(x)/g(x)$.

FOR FURTHER INFORMATION To enhance your understanding of the necessity of the restriction that $g'(x)$ be nonzero for all x in (a, b), except possibly at c, see the article "Counterexamples to L'Hôpital's Rule" by R. P. Boas in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

L'Hôpital's Rule can also be applied to one-sided limits. For instance, if the limit of $f(x)/g(x)$ as x approaches c *from the right* produces the indeterminate form $0/0$, then

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

TECHNOLOGY *Numerical and Graphical Approaches* Use a numerical or a graphical approach to approximate each limit.

a. $\lim_{x\to 0} \frac{2^{2x} - 1}{x}$

b. $\lim_{x\to 0} \frac{3^{2x} - 1}{x}$

c. $\lim_{x\to 0} \frac{4^{2x} - 1}{x}$

d. $\lim_{x\to 0} \frac{5^{2x} - 1}{x}$

What pattern do you observe? Does an analytic approach have an advantage for these limits? If so, explain your reasoning.

EXAMPLE 1 Indeterminate Form 0/0

Evaluate $\lim_{x\to 0} \frac{e^{2x} - 1}{x}$.

Solution Because direct substitution results in the indeterminate form 0/0

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} \qquad \lim_{x\to 0} (e^{2x} - 1) = 0 \qquad \lim_{x\to 0} x = 0$$

you can apply L'Hôpital's Rule as shown below.

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[e^{2x} - 1]}{\frac{d}{dx}[x]}$$ Apply L'Hôpital's Rule.

$$= \lim_{x\to 0} \frac{2e^{2x}}{1}$$ Differentiate numerator and denominator.

$$= 2$$ Evaluate the limit.

NOTE In writing the string of equations in Example 1, you actually do not know that the first limit is equal to the second until you have shown that the second limit exists. In other words, if the second limit had not existed, it would not have been permissible to apply L'Hôpital's Rule.

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ (or $-\infty$) produces the indeterminate form 0/0 or ∞/∞, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

EXAMPLE 2 Indeterminate Form ∞/∞

Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]}$$ Apply L'Hôpital's Rule.

$$= \lim_{x\to\infty} \frac{1}{x}$$ Differentiate numerator and denominator.

$$= 0.$$ Evaluate the limit.

NOTE Try graphing $y_1 = \ln x$ and $y_2 = x$ in the same viewing window. Which function grows faster as x approaches ∞? How is this observation related to Example 2?

Occasionally it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form, as shown in Example 3.

EXAMPLE 3 Applying L'Hôpital's Rule More Than Once

Evaluate $\lim_{x\to-\infty} \dfrac{x^2}{e^{-x}}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule.

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

This limit yields the indeterminate form $(-\infty)/(-\infty)$, so you can apply L'Hôpital's Rule again to obtain

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[-e^{-x}]} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0.$$

In addition to the forms $0/0$ and ∞/∞, there are other indeterminate forms such as $0 \cdot \infty$, 1^∞, ∞^0, 0^0, and $\infty - \infty$. For example, consider the following four limits that lead to the indeterminate form $0 \cdot \infty$.

$$\underbrace{\lim_{x\to0}(x)\left(\frac{1}{x}\right)}_{\text{Limit is 1.}},\quad \underbrace{\lim_{x\to0}(x)\left(\frac{2}{x}\right)}_{\text{Limit is 2.}},\quad \underbrace{\lim_{x\to\infty}(x)\left(\frac{1}{e^x}\right)}_{\text{Limit is 0.}},\quad \underbrace{\lim_{x\to\infty}(e^x)\left(\frac{1}{x}\right)}_{\text{Limit is }\infty.}$$

Because each limit is different, it is clear that the form $0 \cdot \infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit. The following examples indicate methods for evaluating these forms. Basically, you attempt to convert each of these forms to $0/0$ or ∞/∞ so that L'Hôpital's Rule can be applied.

EXAMPLE 4 Indeterminate Form $0 \cdot \infty$

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

Solution Because direct substitution produces the indeterminate form $0 \cdot \infty$, you should try to rewrite the limit to fit the form $0/0$ or ∞/∞. In this case, you can rewrite the limit to fit the second form.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$$

Now, by L'Hôpital's Rule, you have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}\,e^x} = 0.$$

If rewriting a limit in one of the forms $0/0$ or ∞/∞ does not seem to work, try the other form. For instance, in Example 4 you can write the limit as

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}}$$

which yields the indeterminate form $0/0$. As it happens, applying L'Hôpital's Rule to this limit produces

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}} = \lim_{x\to\infty} \frac{-e^{-x}}{-1/(2x^{3/2})}$$

which also yields the indeterminate form $0/0$.

The indeterminate forms 1^∞, ∞^0, and 0^0 arise from limits of functions that have variable bases and variable exponents. When you previously encountered this type of function, you used logarithmic differentiation to find the derivative. You can use a similar procedure when taking limits, as shown in the next example.

EXAMPLE 5 Indeterminate Form 1^∞

Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution Because direct substitution yields the indeterminate form 1^∞, you can proceed as follows. To begin, assume that the limit exists and is equal to y.

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Taking the natural logarithm of each side produces

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right].$$

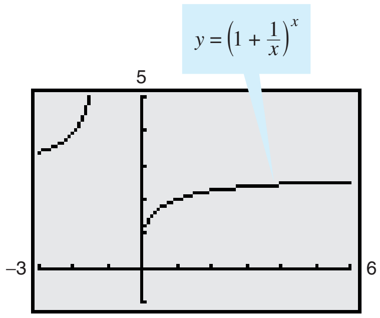
Because the natural logarithmic function is continuous, you can write

$$\begin{aligned}
\ln y &= \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] && \text{Indeterminate form } \infty\cdot 0\\
&= \lim_{x\to\infty}\left(\frac{\ln[1+(1/x)]}{1/x}\right) && \text{Indeterminate form } 0/0\\
&= \lim_{x\to\infty}\left(\frac{(-1/x^2)\{1/[1+(1/x)]\}}{-1/x^2}\right) && \text{L'Hôpital's Rule}\\
&= \lim_{x\to\infty}\frac{1}{1+(1/x)}\\
&= 1.
\end{aligned}$$

Now, because you have shown that $\ln y = 1$, you can conclude that $y = e$ and obtain

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

You can use a graphing utility to confirm this result, as shown in Figure 8.15.



The limit of $[1+(1/x)]^x$ as x approaches infinity is e.
Figure 8.15

L'Hôpital's Rule can also be applied to one-sided limits, as demonstrated in Examples 6 and 7.

EXAMPLE 6 Indeterminate Form 0^0

Find $\lim_{x\to 0^+} (\sin x)^x$.

Solution Because direct substitution produces the indeterminate form 0^0, you can proceed as shown below. To begin, assume that the limit exists and is equal to y.

$$y = \lim_{x\to 0^+} (\sin x)^x \qquad \text{Indeterminate form } 0^0$$

$$\ln y = \ln\left[\lim_{x\to 0^+} (\sin x)^x\right] \qquad \text{Take natural log of each side.}$$

$$= \lim_{x\to 0^+} [\ln(\sin x)^x] \qquad \text{Continuity}$$

$$= \lim_{x\to 0^+} [x \ln(\sin x)] \qquad \text{Indeterminate form } 0 \cdot (-\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} \qquad \text{Indeterminate form } -\infty/\infty$$

$$= \lim_{x\to 0^+} \frac{\cot x}{-1/x^2} \qquad \text{L'Hôpital's Rule}$$

$$= \lim_{x\to 0^+} \frac{-x^2}{\tan x} \qquad \text{Indeterminate form } 0/0$$

$$= \lim_{x\to 0^+} \frac{-2x}{\sec^2 x} = 0 \qquad \text{L'Hôpital's Rule}$$

Now, because $\ln y = 0$, you can conclude that $y = e^0 = 1$, and it follows that

$$\lim_{x\to 0^+} (\sin x)^x = 1.$$

TECHNOLOGY When evaluating complicated limits such as the one in Example 6, it is helpful to check the reasonableness of the solution with a computer or with a graphing utility. For instance, the calculations in the following table and the graph in Figure 8.16 are consistent with the conclusion that $(\sin x)^x$ approaches 1 as x approaches 0 from the right.

x	1.0	0.1	0.01	0.001	0.0001	0.00001
$(\sin x)^x$	0.8415	0.7942	0.9550	0.9931	0.9991	0.9999

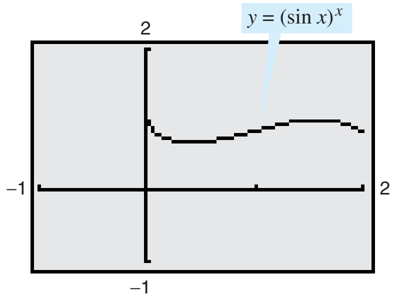
Use a computer algebra system or graphing utility to estimate the following limits:

$$\lim_{x\to 0} (1 - \cos x)^x$$

and

$$\lim_{x\to 0^+} (\tan x)^x.$$

Then see if you can verify your estimates analytically.



The limit of $(\sin x)^x$ is 1 as x approaches 0 from the right.
Figure 8.16

STUDY TIP In each of the examples presented in this section, L'Hôpital's Rule is used to find a limit that exists. It can also be used to conclude that a limit is infinite. For instance, try using L'Hôpital's Rule to show that

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty.$$

EXAMPLE 7 Indeterminate Form $\infty - \infty$

Evaluate $\displaystyle\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Solution Because direct substitution yields the indeterminate form $\infty - \infty$, you should try to rewrite the expression to produce a form to which you can apply L'Hôpital's Rule. In this case, you can combine the two fractions to obtain

$$\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \left[\frac{x-1-\ln x}{(x-1)\ln x}\right].$$

Now, because direct substitution produces the indeterminate form $0/0$, you can apply L'Hôpital's Rule to obtain

$$\begin{aligned}\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to 1^+} \frac{\frac{d}{dx}[x-1-\ln x]}{\frac{d}{dx}[(x-1)\ln x]}\\ &= \lim_{x\to 1^+} \left[\frac{1-(1/x)}{(x-1)(1/x)+\ln x}\right]\\ &= \lim_{x\to 1^+} \left(\frac{x-1}{x-1+x\ln x}\right).\end{aligned}$$

This limit also yields the indeterminate form $0/0$, so you can apply L'Hôpital's Rule again to obtain

$$\begin{aligned}\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to 1^+} \left[\frac{1}{1+x(1/x)+\ln x}\right]\\ &= \frac{1}{2}.\end{aligned}$$

The forms $0/0$, ∞/∞, $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0 have been identified as *indeterminate*. There are similar forms that you should recognize as "determinate."

$\infty + \infty \to \infty$	Limit is positive infinity.
$-\infty - \infty \to -\infty$	Limit is negative infinity.
$0^\infty \to 0$	Limit is zero.
$0^{-\infty} \to \infty$	Limit is positive infinity.

(You are asked to verify two of these in Exercises 106 and 107.)

As a final comment, remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $0/0$ and ∞/∞. For instance, the following application of L'Hôpital's Rule is *incorrect*.

$$\lim_{x\to 0} \frac{e^x}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$$

Incorrect use of L'Hôpital's Rule

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1, which means that the hypotheses of L'Hôpital's Rule have not been satisfied.

Exercises for Section 8.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Numerical and Graphical Analysis **In Exercises 1–4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to support your result.**

1. $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 2x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

2. $\lim_{x\to 0} \dfrac{1 - e^x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

3. $\lim_{x\to\infty} x^5 e^{-x/100}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

4. $\lim_{x\to\infty} \dfrac{6x}{\sqrt{3x^2 - 2x}}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

In Exercises 5–10, evaluate the limit (a) using techniques from Chapters 1 and 3 and (b) using L'Hôpital's Rule.

5. $\lim_{x\to 3} \dfrac{2(x-3)}{x^2 - 9}$

6. $\lim_{x\to -1} \dfrac{2x^2 - x - 3}{x + 1}$

7. $\lim_{x\to 3} \dfrac{\sqrt{x+1} - 2}{x - 3}$

8. $\lim_{x\to 0} \dfrac{\sin 4x}{2x}$

9. $\lim_{x\to\infty} \dfrac{5x^2 - 3x + 1}{3x^2 - 5}$

10. $\lim_{x\to\infty} \dfrac{2x + 1}{4x^2 + x}$

In Exercises 11–36, evaluate the limit, using L'Hôpital's Rule if necessary. (In Exercise 18, *n* is a positive integer.)

11. $\lim_{x\to 2} \dfrac{x^2 - x - 2}{x - 2}$

12. $\lim_{x\to -1} \dfrac{x^2 - x - 2}{x + 1}$

13. $\lim_{x\to 0} \dfrac{\sqrt{4 - x^2} - 2}{x}$

14. $\lim_{x\to 2^-} \dfrac{\sqrt{4 - x^2}}{x - 2}$

15. $\lim_{x\to 0} \dfrac{e^x - (1 - x)}{x}$

16. $\lim_{x\to 1} \dfrac{\ln x^2}{x^2 - 1}$

17. $\lim_{x\to 0^+} \dfrac{e^x - (1 + x)}{x^3}$

18. $\lim_{x\to 0^+} \dfrac{e^x - (1 + x)}{x^n}$

19. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

20. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$

21. $\lim_{x\to 0} \dfrac{\arcsin x}{x}$

22. $\lim_{x\to 1} \dfrac{\arctan x - (\pi/4)}{x - 1}$

23. $\lim_{x\to\infty} \dfrac{3x^2 - 2x + 1}{2x^2 + 3}$

24. $\lim_{x\to\infty} \dfrac{x - 1}{x^2 + 2x + 3}$

25. $\lim_{x\to\infty} \dfrac{x^2 + 2x + 3}{x - 1}$

26. $\lim_{x\to\infty} \dfrac{x^3}{x + 2}$

27. $\lim_{x\to\infty} \dfrac{x^3}{e^{x/2}}$

28. $\lim_{x\to\infty} \dfrac{x^2}{e^x}$

29. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 + 1}}$

30. $\lim_{x\to\infty} \dfrac{x^2}{\sqrt{x^2 + 1}}$

31. $\lim_{x\to\infty} \dfrac{\cos x}{x}$

32. $\lim_{x\to\infty} \dfrac{\sin x}{x - \pi}$

33. $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$

34. $\lim_{x\to\infty} \dfrac{\ln x^4}{x^3}$

35. $\lim_{x\to\infty} \dfrac{e^x}{x^2}$

36. $\lim_{x\to\infty} \dfrac{e^{x/2}}{x}$

In Exercises 37–54, (a) describe the type of indeterminate form (if any) that is obtained by direct substitution. (b) Evaluate the limit, using L'Hôpital's Rule if necessary. (c) Use a graphing utility to graph the function and verify the result in part (b).

37. $\lim_{x\to\infty} x \ln x$

38. $\lim_{x\to 0^+} x^3 \cot x$

39. $\lim_{x\to\infty} \left(x \sin \dfrac{1}{x}\right)$

40. $\lim_{x\to\infty} x \tan \dfrac{1}{x}$

41. $\lim_{x\to 0^+} x^{1/x}$

42. $\lim_{x\to 0^+} (e^x + x)^{2/x}$

43. $\lim_{x\to\infty} x^{1/x}$

44. $\lim_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x$

45. $\lim_{x\to 0^+} (1 + x)^{1/x}$

46. $\lim_{x\to\infty} (1 + x)^{1/x}$

47. $\lim_{x\to 0^+} [3(x)^{x/2}]$

48. $\lim_{x\to 4^+} [3(x - 4)]^{x-4}$

49. $\lim_{x\to 1^+} (\ln x)^{x-1}$

50. $\lim_{x\to 0^+} \left[\cos\left(\dfrac{\pi}{2} - x\right)\right]^x$

51. $\lim_{x\to 2^+} \left(\dfrac{8}{x^2 - 4} - \dfrac{x}{x - 2}\right)$

52. $\lim_{x\to 2^+} \left(\dfrac{1}{x^2 - 4} - \dfrac{\sqrt{x - 1}}{x^2 - 4}\right)$

53. $\lim_{x\to 1^+} \left(\dfrac{3}{\ln x} - \dfrac{2}{x - 1}\right)$

54. $\lim_{x\to 0^+} \left(\dfrac{10}{x} - \dfrac{3}{x^2}\right)$

In Exercises 55–58, use a graphing utility to (a) graph the function and (b) find the required limit (if it exists).

55. $\lim_{x\to 3} \dfrac{x - 3}{\ln(2x - 5)}$

56. $\lim_{x\to 0^+} (\sin x)^x$

57. $\lim_{x\to\infty} \left(\sqrt{x^2 + 5x + 2} - x\right)$

58. $\lim_{x\to\infty} \dfrac{x^3}{e^{2x}}$

Writing About Concepts

59. List six different indeterminate forms.

60. State L'Hôpital's Rule.

61. Find the differentiable functions f and g that satisfy the specified condition such that

$$\lim_{x\to 5} f(x) = 0 \text{ and } \lim_{x\to 5} g(x) = 0.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

(a) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 10$ (b) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \infty$

62. Find differentiable functions f and g such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \text{and}$$

$$\lim_{x\to\infty} [f(x) - g(x)] = 25.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

63. ***Numerical Approach*** Complete the table to show that x eventually "overpowers" $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$						

64. ***Numerical Approach*** Complete the table to show that e^x eventually "overpowers" x^5.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

Comparing Functions **In Exercises 65–70, use L'Hôpital's Rule to determine the comparative rates of increase of the functions**

$$f(x) = x^m, \quad g(x) = e^{nx}, \quad \text{and} \quad h(x) = (\ln x)^n$$

where $n > 0$, $m > 0$, and $x \to \infty$.

65. $\lim_{x\to\infty} \frac{x^2}{e^{5x}}$

66. $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$

67. $\lim_{x\to\infty} \frac{(\ln x)^3}{x}$

68. $\lim_{x\to\infty} \frac{(\ln x)^2}{x^3}$

69. $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$

70. $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

In Exercises 71–74, find any asymptotes and relative extrema that may exist and use a graphing utility to graph the function. (*Hint:* Some of the limits required in finding asymptotes have been found in preceding exercises.)

71. $y = x^{1/x}, \quad x > 0$

72. $y = x^x, \quad x > 0$

73. $y = 2xe^{-x}$

74. $y = \frac{\ln x}{x}$

Think About It **In Exercises 75–78, L'Hôpital's Rule is used incorrectly. Describe the error.**

75. $\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \lim_{x\to 0} \frac{2e^{2x}}{e^x} = \lim_{x\to 0} 2e^x = 2$

76. $\lim_{x\to 0} \frac{\sin \pi x - 1}{x} = \lim_{x\to 0} \frac{\pi \cos \pi x}{1} = \pi$

77. $\lim_{x\to\infty} x \cos \frac{1}{x} = \lim_{x\to\infty} \frac{\cos(1/x)}{1/x}$

$$= \lim_{x\to\infty} \frac{[-\sin(1/x)](1/x^2)}{-1/x^2}$$

$$= 0$$

78. $\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \lim_{x\to\infty} \frac{-e^{-x}}{-e^{-x}}$

$$= \lim_{x\to\infty} 1$$

$$= 1$$

Analytical Approach **In Exercises 79 and 80, (a) explain why L'Hôpital's Rule cannot be used to find the limit, (b) find the limit analytically, and (c) use a graphing utility to graph the function and approximate the limit from the graph. Compare the result with that in part (b).**

79. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}$

80. $\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x}$

Graphical Analysis **In Exercises 81 and 82, graph $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$. What do you notice about these ratios as $x \to 0$? How does this illustrate L'Hôpital's Rule?**

81. $f(x) = \sin 3x, \quad g(x) = \sin 4x$

82. $f(x) = e^{3x} - 1, \quad g(x) = x$

83. ***Velocity in a Resisting Medium*** The velocity v of an object falling through a resisting medium such as air or water is given by

$$v = \frac{32}{k}\left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32}\right)$$

where v_0 is the initial velocity, t is the time in seconds, and k is the resistance constant of the medium. Use L'Hôpital's Rule to find the formula for the velocity of a falling body in a vacuum by fixing v_0 and t and letting k approach zero. (Assume that the downward direction is positive.)

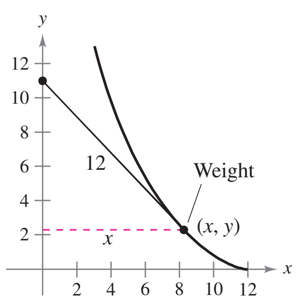
84. ***Compound Interest*** The formula for the amount A in a savings account compounded n times per year for t years at an interest rate r and an initial deposit of P is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as the number of compoundings per year becomes infinite is given by $A = Pe^{rt}$.

85. ***The Gamma Function*** The Gamma Function $\Gamma(n)$ is defined in terms of the integral of the function given by $f(x) = x^{n-1}e^{-x}$, $n > 0$. Show that for any fixed value of n, the limit of $f(x)$ as x approaches infinity is zero.

86. ***Tractrix*** A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope (see figure). Initially, the weight is located at the point $(12, 0)$.



(a) Show that the slope of the tangent line of the path of the weight is

$$\frac{dy}{dx} = -\frac{\sqrt{144 - x^2}}{x}.$$

(b) Use the result of part (a) to find the equation of the path of the weight. Use a graphing utility to graph the path and compare it with the figure.

(c) Find any vertical asymptotes of the graph in part (b).

(d) When the person has reached the point $(0, 12)$, how far has the weight moved?

In Exercises 87–90, apply the Extended Mean Value Theorem to the functions f and g on the given interval. Find all values c in the interval (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

	Functions	*Interval*
87.	$f(x) = x^3, \quad g(x) = x^2 + 1$	$[0, 1]$
88.	$f(x) = \frac{1}{x}, \quad g(x) = x^2 - 4$	$[1, 2]$
89.	$f(x) = \sin x, \quad g(x) = \cos x$	$\left[0, \frac{\pi}{2}\right]$
90.	$f(x) = \ln x, \quad g(x) = x^3$	$[1, 4]$

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

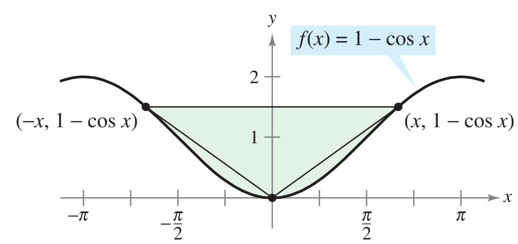
91. $\lim_{x\to 0}\left[\frac{x^2 + x + 1}{x}\right] = \lim_{x\to 0}\left[\frac{2x + 1}{1}\right] = 1$

92. If $y = e^x/x^2$, then $y' = e^x/2x$.

93. If $p(x)$ is a polynomial, then $\lim_{x\to\infty}\left[p(x)/e^x\right] = 0$.

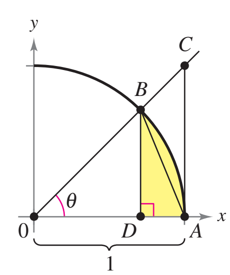
94. If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$, then $\lim_{x\to\infty}\left[f(x) - g(x)\right] = 0$.

95. ***Area*** Find the limit, as x approaches 0, of the ratio of the area of the triangle to the total shaded area in the figure.



96. In Section 1.3, a geometric argument (see figure) was used to prove that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$



(a) Write the area of $\triangle ABD$ in terms of θ.

(b) Write the area of the shaded region in terms of θ.

(c) Write the ratio R of the area of $\triangle ABD$ to that of the shaded region.

(d) Find $\lim_{\theta\to 0} R$.

Continuous Functions **In Exercises 97 and 98, find the value of c that makes the function continuous at $x = 0$.**

97. $f(x) = \begin{cases} \dfrac{4x - 2\sin 2x}{2x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$

98. $f(x) = \begin{cases} (e^x + x)^{1/x}, & x \neq 0 \\ c, & x = 0 \end{cases}$

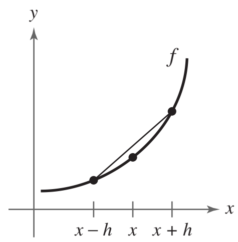
99. Find the values of a and b such that $\lim_{x\to 0}\frac{a - \cos bx}{x^2} = 2$.

100. Show that $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$ for any integer $n > 0$.

101. (a) Let $f'(x)$ be continuous. Show that

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}=f'(x).$$

(b) Explain the result of part (a) graphically.



102. Let $f''(x)$ be continuous. Show that

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x).$$

103. Sketch the graph of

$$g(x)=\begin{cases} e^{-1/x^2}, & x\neq 0\\ 0, & x=0\end{cases}$$

and determine $g'(0)$.

104. Use a graphing utility to graph

$$f(x)=\frac{x^k-1}{k}$$

for $k = 1, 0.1,$ and 0.01. Then evaluate the limit

$$\lim_{k\to 0^+}\frac{x^k-1}{k}.$$

105. Consider the limit $\lim_{x\to 0^+}(-x\ln x)$.

(a) Describe the type of indeterminate form that is obtained by direct substitution.

(b) Evaluate the limit.

(c) Use a graphing utility to verify the result of part (b).

FOR FURTHER INFORMATION For a geometric approach to this exercise, see the article "A Geometric Proof of $\lim_{d\to 0^+}(-d\ln d)=0$" by John H. Mathews in the *College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

106. Prove that if $f(x)\geq 0$, $\lim_{x\to a} f(x)=0$, and $\lim_{x\to a} g(x)=\infty$, then $\lim_{x\to a} f(x)^{g(x)}=0$.

107. Prove that if $f(x)\geq 0$, $\lim_{x\to a} f(x)=0$, and $\lim_{x\to a} g(x)=-\infty$, then $\lim_{x\to a} f(x)^{g(x)}=\infty$.

108. Prove the following generalization of the Mean Value Theorem. If f is twice differentiable on the closed interval $[a, b]$, then

$$f(b)-f(a)=f'(a)(b-a)-\int_a^b f''(t)(t-b)\,dt.$$

109. ***Indeterminate Forms*** Show that the indeterminate forms 0^0, ∞^0, and 1^∞ do not always have a value of 1 by evaluating each limit.

(a) $\lim_{x\to 0^+} x^{\ln 2/(1+\ln x)}$

(b) $\lim_{x\to\infty} x^{\ln 2/(1+\ln x)}$

(c) $\lim_{x\to 0}(x+1)^{(\ln 2)/x}$

110. ***Calculus History*** In L'Hôpital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x)=\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$$

as x approaches a, $a > 0$. Find this limit.

111. Consider the function

$$h(x)=\frac{x+\sin x}{x}.$$

(a) Use a graphing utility to graph the function. Then use the *zoom* and *trace* features to investigate $\lim_{x\to\infty} h(x)$.

(b) Find $\lim_{x\to\infty} h(x)$ analytically by writing

$$h(x)=\frac{x}{x}+\frac{\sin x}{x}.$$

(c) Can you use L'Hôpital's Rule to find $\lim_{x\to\infty} h(x)$? Explain your reasoning.

Putnam Exam Challenge

112. Evaluate

$$\lim_{x\to\infty}\left[\frac{1}{x}\cdot\frac{a^x-1}{a-1}\right]^{1/x}$$

where $a > 0$, $a \neq 1$.

This problem was composed by the Committee on the Putnam Prize Competition.