Newton's Method

In this section you will study a technique for approximating the real zeros of a function. The technique is called Newton’s Method, and it uses tangent lines to approximate the graph of the function near its x-intercepts.

To see how Newton’s Method works, consider a function $f$ that is continuous on the interval $[a, b]$ and differentiable on the interval $(a, b)$. If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem, $f$ must have at least one zero in the interval $(a, b)$. Suppose you estimate this zero to occur at $x = x_1$ as shown in Figure 3.60(a). Newton’s Method is based on the assumption that the graph of $f$ and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at about the same point. Because you can easily calculate the x-intercept for this tangent line, you can use it as a second (and, usually, better) estimate for the zero of $f$. The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, the equation of the tangent line is therefore

$$y - f(x_1) = f'(x_1)(x - x_1).$$

Letting $y = 0$ and solving for $x$ produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$  

So, from the initial estimate $x_1$ you obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$  

You can improve on $x_2$ and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$  

Repeated application of this process is called Newton’s Method.

**Newton’s Method for Approximating the Zeros of a Function**

Let $f(c) = 0$, where $f$ is differentiable on an open interval containing $c$. Then, to approximate $c$, use the following steps.

1. Make an initial estimate $x_1$ that is close to $c$. (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$  

3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let $x_{n+1}$ serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an iteration.
CHAPTER 3 Applications of Differentiation

EXAMPLE 1 Using Newton's Method

Calculate three iterations of Newton’s Method to approximate a zero of \( f(x) = x^2 - 2 \). Use \( x_1 = 1 \) as the initial guess.

Solution Because \( f(x) = x^2 - 2 \), you have \( f'(x) = 2x \), and the iterative process is given by the formula

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}.
\]

The calculations for three iterations are shown in the table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( f(x_n) )</th>
<th>( f'(x_n) )</th>
<th>( \frac{f(x_n)}{f'(x_n)} )</th>
<th>( x_n - \frac{f(x_n)}{f'(x_n)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00000</td>
<td>-1.00000</td>
<td>2.00000</td>
<td>-0.500000</td>
<td>1.500000</td>
</tr>
<tr>
<td>2</td>
<td>1.50000</td>
<td>0.250000</td>
<td>3.00000</td>
<td>0.083333</td>
<td>1.416667</td>
</tr>
<tr>
<td>3</td>
<td>1.416667</td>
<td>0.006945</td>
<td>2.833334</td>
<td>0.002451</td>
<td>1.414216</td>
</tr>
<tr>
<td>4</td>
<td>1.414216</td>
<td>0.00001</td>
<td>5.66533</td>
<td>0.00000</td>
<td>1.414216</td>
</tr>
</tbody>
</table>

Of course, in this case you know that the two zeros of the function are \( \pm \sqrt{2} \). To six decimal places, \( \sqrt{2} = 1.414214 \). So, after only three iterations of Newton’s Method, you have obtained an approximation that is within 0.000002 of an actual root. The first iteration of this process is shown in Figure 3.61.

EXAMPLE 2 Using Newton's Method

Use Newton’s Method to approximate the zeros of

\[
f(x) = 2x^3 + x^2 - x + 1.
\]

Continue the iterations until two successive approximations differ by less than 0.0001.

Solution Begin by sketching a graph of \( f \), as shown in Figure 3.62. From the graph, you can observe that the function has only one zero, which occurs near \( x = -1.2 \).

Next, differentiate \( f \) and form the iterative formula

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n^3 + x_n^2 - x_n + 1}{6x_n^2 + 2x_n - 1}.
\]

The calculations are shown in the table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( f(x_n) )</th>
<th>( f'(x_n) )</th>
<th>( \frac{f(x_n)}{f'(x_n)} )</th>
<th>( x_n - \frac{f(x_n)}{f'(x_n)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.20000</td>
<td>0.184000</td>
<td>5.240000</td>
<td>0.03511</td>
<td>-1.23511</td>
</tr>
<tr>
<td>2</td>
<td>-1.23511</td>
<td>-0.00771</td>
<td>5.68276</td>
<td>-0.00136</td>
<td>-1.23375</td>
</tr>
<tr>
<td>3</td>
<td>-1.23375</td>
<td>0.00001</td>
<td>5.66533</td>
<td>0.00000</td>
<td>-1.23375</td>
</tr>
<tr>
<td>4</td>
<td>-1.23375</td>
<td>0.00000</td>
<td>5.66533</td>
<td>0.00000</td>
<td>-1.23375</td>
</tr>
</tbody>
</table>

Because two successive approximations differ by less than the required 0.0001, you can estimate the zero of \( f \) to be \(-1.23375\).
When, as in Examples 1 and 2, the approximations approach a limit, the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ is said to **converge**. Moreover, if the limit is $c$, it can be shown that $c$ must be a zero of $f$.

Newton’s Method does not always yield a convergent sequence. One way it can fail to do so is shown in Figure 3.63. Because Newton’s Method involves division by $f'(x_n)$, it is clear that the method will fail if the derivative is zero for any $x_n$ in the sequence. When you encounter this problem, you can usually overcome it by choosing a different value for $x_1$. Another way Newton’s Method can fail is shown in the next example.

**EXAMPLE 3**  **An Example in Which Newton’s Method Fails**

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton’s Method fails to converge using $x_1 = 0.1$.

**Solution** Because $f'(x) = \frac{1}{3}x^{-2/3}$, the iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$$

$$= x_n - 3x_n$$

$$= -2x_n.$$

The calculations are shown in the table. This table and Figure 3.64 indicate that $x_n$ continues to increase in magnitude as $n \to \infty$, and so the limit of the sequence does not exist.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$f(x_n)$</th>
<th>$f'(x_n)$</th>
<th>$\frac{f(x_n)}{f'(x_n)}$</th>
<th>$x_n - \frac{f(x_n)}{f'(x_n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10000</td>
<td>0.46416</td>
<td>1.54720</td>
<td>0.30000</td>
<td>-0.20000</td>
</tr>
<tr>
<td>2</td>
<td>-0.20000</td>
<td>-0.58480</td>
<td>0.97467</td>
<td>-0.60000</td>
<td>0.40000</td>
</tr>
<tr>
<td>3</td>
<td>0.40000</td>
<td>0.73681</td>
<td>0.61401</td>
<td>1.20000</td>
<td>-0.80000</td>
</tr>
<tr>
<td>4</td>
<td>-0.80000</td>
<td>-0.92832</td>
<td>0.38680</td>
<td>-2.40000</td>
<td>1.60000</td>
</tr>
</tbody>
</table>

**NOTE** In Example 3, the initial estimate $x_1 = 0.1$ fails to produce a convergent sequence. Try showing that Newton’s Method also fails for every other choice of $x_1$ (other than the actual zero).
It can be shown that a condition sufficient to produce convergence of Newton’s Method to a zero of \( f \) is that
\[
\left| \frac{f(x) f''(x)}{(f'(x))^2} \right| < 1
\]
Condition for convergence

on an open interval containing the zero. For instance, in Example 1 this test would yield \( f(x) = x^2 - 2, f'(x) = 2x, f''(x) = 2 \), and
\[
\left| \frac{f(x) f''(x)}{(f'(x))^2} \right| = \left| \frac{(x^2 - 2)(2)}{4x^2} \right| = \left| \frac{1}{2} - \frac{1}{x^2} \right|.
\]
Example 1

On the interval \((1, 3)\), this quantity is less than 1 and therefore the convergence of Newton’s Method is guaranteed. On the other hand, in Example 3, you have
\[
f(x) = x^{1/3}, f'(x) = \frac{1}{3}x^{-2/3}, f''(x) = -\frac{2}{9}x^{-5/3},
\]
and
\[
\left| \frac{f(x) f''(x)}{(f'(x))^2} \right| = \left| \frac{x^{4/3}(-2/9)(x^{-5/3})}{(1/9)(x^{-2/3})} \right| = 2
\]
Example 3

which is not less than 1 for any value of \( x \), so you cannot conclude that Newton’s Method will converge.

### Algebraic Solutions of Polynomial Equations

The zeros of some functions, such as
\[
f(x) = x^3 - 2x^2 - x + 2
\]
can be found by simple algebraic techniques, such as factoring. The zeros of other functions, such as
\[
f(x) = x^3 - x + 1
\]
cannot be found by elementary algebraic methods. This particular function has only one real zero, and by using more advanced algebraic techniques you can determine the zero to be
\[
x = -\sqrt[3]{\frac{3 - \sqrt{23/3}}{6}} - \sqrt[3]{\frac{3 + \sqrt{23/3}}{6}}.
\]

Because the exact solution is written in terms of square roots and cube roots, it is called a solution by radicals.

**NOTE** Try approximating the real zero of \( f(x) = x^3 - x + 1 \) and compare your result with the exact solution shown above.

The determination of radical solutions of a polynomial equation is one of the fundamental problems of algebra. The earliest such result is the Quadratic Formula, which dates back at least to Babylonian times. The general formula for the zeros of a cubic function was developed much later. In the sixteenth century an Italian mathematician, Jerome Cardan, published a method for finding radical solutions to cubic and quartic equations. Then, for 300 years, the problem of finding a general quintic formula remained open. Finally, in the nineteenth century, the problem was answered independently by two young mathematicians. Niels Henrik Abel, a Norwegian mathematician, and Evariste Galois, a French mathematician, proved that it is not possible to solve a general fifth- (or higher-) degree polynomial equation by radicals. Of course, you can solve particular fifth-degree equations such as \( x^5 - 1 = 0 \), but Abel and Galois were able to show that no general radical solution exists.
In Exercises 1–4, complete two iterations of Newton’s Method for the function using the given initial guess.
1. \( f(x) = x^3 - 3 \), \( x_1 = 1.7 \)
2. \( f(x) = 2x^2 - 3 \), \( x_1 = 1 \)
3. \( f(x) = \sin x \), \( x_1 = 3 \)
4. \( f(x) = \tan x \), \( x_1 = 0.1 \)

In Exercises 5–14, approximate the zero(s) of the function. Use Newton’s Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.
5. \( f(x) = x^3 + x - 1 \)
6. \( f(x) = x^3 + x - 1 \)
7. \( f(x) = 3\sqrt{x} - 3 - x \)
8. \( f(x) = x - 2\sqrt{x} + 1 \)
9. \( f(x) = x^6 + 3 \)
10. \( f(x) = x - 1 - 2x^3 \)
11. \( f(x) = x^3 - 3.9x^2 + 4.79x - 1.881 \)
12. \( f(x) = \frac{1}{2}x^4 - 3x - 3 \)
13. \( f(x) = x + \sin(x + 1) \)
14. \( f(x) = x^3 - \cos x \)

In Exercises 15–18, apply Newton’s Method to approximate the zero(s) of the indicated point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. (Hint: Let \( h(x) = f(x) - g(x) \).)
15. \( f(x) = 2x + 1 \)
   \( g(x) = \sqrt{x + 4} \)
16. \( f(x) = 3 - x \)
   \( g(x) = \frac{1}{x^2 + 1} \)
17. \( f(x) = x \)
   \( g(x) = \tan x \)
18. \( f(x) = x^2 \)
   \( g(x) = \cos x \)

19. **Mechanic’s Rule**  The Mechanic’s Rule for approximating \( \sqrt{a}, a > 0 \), is
   \[ x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right), \quad n = 1, 2, 3 \ldots \]
   where \( x_1 \) is an approximation of \( \sqrt{a} \).

(a) Use Newton’s Method and the function \( f(x) = x^2 - a \) to derive the Mechanic’s Rule.
(b) Use the Mechanic’s Rule to approximate \( \sqrt{3} \) and \( \sqrt{7} \) to three decimal places.

20. (a) Use Newton’s Method and the function \( f(x) = x^e - a \) to obtain a general rule for approximating \( x = \sqrt[a]{a} \).
(b) Use the general rule found in part (a) to approximate \( \sqrt[5]{3} \) and \( \sqrt[13]{5} \) to three decimal places.

In Exercises 21–24, apply Newton’s Method using the given initial guess, and explain why the method fails.
21. \( f(x) = 2x^3 - 6x^2 + 6x - 1 \), \( x_1 = 1 \)
22. \( f(x) = 4x^3 - 12x^2 + 12x - 3 \), \( x_1 = \frac{3}{2} \)

23. \( f(x) = -x^3 + 6x^2 - 10x + 6 \), \( x_1 = 2 \)
24. \( f(x) = 2 \sin x + \cos 2x \), \( x_1 = \frac{3\pi}{2} \)

25. In your own words and using a sketch, describe Newton’s Method for approximating the zeros of a function.
26. Under what conditions will Newton’s Method fail?

**Fixed Points**  In Exercises 27 and 28, approximate the fixed point of the function to two decimal places. [A fixed point \( x_0 \) of a function \( f \) is a value of \( x \) such that \( f(x_0) = x_0 \).]
27. \( f(x) = \cos x \)
28. \( f(x) = \cot x \), \( 0 < x < \pi \)
29. **Writing** Consider the function \( f(x) = x^3 - 3x^2 + 3 \).

(a) Use a graphing utility to graph \( f \).

(b) Use Newton’s Method with \( x_1 = 1 \) as an initial guess.

(c) Repeat part (b) using \( x_1 = \frac{1}{2} \) as an initial guess and observe that the result is different.

(d) To understand why the results in parts (b) and (c) are different, sketch the tangent lines to the graph of \( f \) at the points \( (1, f(1)) \) and \( \left( \frac{1}{2}, f(\frac{1}{2}) \right) \). Find the \( x \)-intercept of each tangent line and compare the intercepts with the first iteration of Newton’s Method using the respective initial guesses.

(e) Write a short paragraph summarizing how Newton’s Method works. Use the results of this exercise to describe why it is important to select the initial guess carefully.

30. **Writing** Repeat the steps in Exercise 29 for the function \( f(x) = \sin x \) with initial guesses of \( x_1 = 1.8 \) and \( x_1 = 3 \).

31. Use Newton’s Method to show that the equation \( x_{n+1} = x_n \left( 2 - ax_n \right) \) can be used to approximate \( 1/a \) if \( x_1 \) is an initial guess of the reciprocal of \( a \). Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. [Hint: Consider \( f(x) = (1/x) - a \).]

32. Use the result of Exercise 31 to approximate \( 1/3 \) and \( 1/11 \) to three decimal places.

In Exercises 33 and 34, approximate the critical number of \( f \) on the interval \((0, \pi)\). Sketch the graph of \( f \), labeling any extrema.

33. \( f(x) = x \cos x \)

34. \( f(x) = x \sin x \)

In Exercises 35–38, some typical problems from the previous sections of this chapter are given. In each case, use Newton’s Method to approximate the solution.

35. **Minimum Distance** Find the point on the graph of \( f(x) = 4 - x^2 \) that is closest to the point \((1, 0)\).

36. **Minimum Distance** Find the point on the graph of \( f(x) = x^2 \) that is closest to the point \((4, -3)\).

37. **Minimum Time** You are in a boat 2 miles from the nearest point on the coast (see figure). You are to go to a point \( Q \), which is 3 miles down the coast and 1 mile inland. You can row at 3 miles per hour and walk at 4 miles per hour. Toward what point on the coast should you row in order to reach \( Q \) in the least time?

38. **Medicine** The concentration \( C \) of a chemical in the bloodstream \( t \) hours after injection into muscle tissue is given by \( C = (3t^2 + t)/(50 + t) \). When is the concentration greatest?

39. **Advertising Costs** A company that produces portable CD players estimates that the profit for selling a particular model is

\[
P = -76x^3 + 4830x^2 - 320,000, \quad 0 \leq x \leq 60
\]

where \( P \) is the profit in dollars and \( x \) is the advertising expense in 10,000s of dollars (see figure). According to this model, find the smaller of two advertising amounts that yield a profit \( P \) of $2,500,000.

![Figure for 39](image)

**Figure for 39**

**40. Engine Power** The torque produced by a compact automobile engine is approximated by the model

\[
T = 0.808x^3 - 17.974x^2 + 71.248x + 110.843, \quad 1 \leq x \leq 5
\]

where \( T \) is the torque in foot-pounds and \( x \) is the engine speed in thousands of revolutions per minute (see figure). Approximate the two engine speeds that yield a torque \( T \) of 170 foot-pounds.

**True or False?** In Exercises 41–44, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

41. The zeros of \( f(x) = p(x)/q(x) \) coincide with the zeros of \( p(x) \).

42. If the coefficients of a polynomial function are all positive, then the polynomial has no positive zeros.

43. If \( f(x) \) is a cubic polynomial such that \( f(x) \) is never zero, then any initial guess will force Newton’s Method to converge to the zero of \( f \).

44. The roots of \( \sqrt{f(x)} = 0 \) coincide with the roots of \( f(x) = 0 \).

45. **Tangent Lines** The graph of \( f(x) = -\sin x \) has infinitely many tangent lines that pass through the origin. Use Newton’s Method to approximate the slope of the tangent line having the greatest slope to three decimal places.

46. Consider the function \( f(x) = 2x^3 - 20x^2 - 12x - 24 \).

(a) Use a graphing utility to determine the number of zeros of \( f \).

(b) Use Newton’s Method with an initial estimate of \( x_1 = 2 \) to approximate the zero of \( f \) to four decimal places.

(c) Repeat part (b) using initial estimates of \( x_1 = 10 \) and \( x_1 = 100 \).

(d) Discuss the results of parts (b) and (c). What can you conclude?